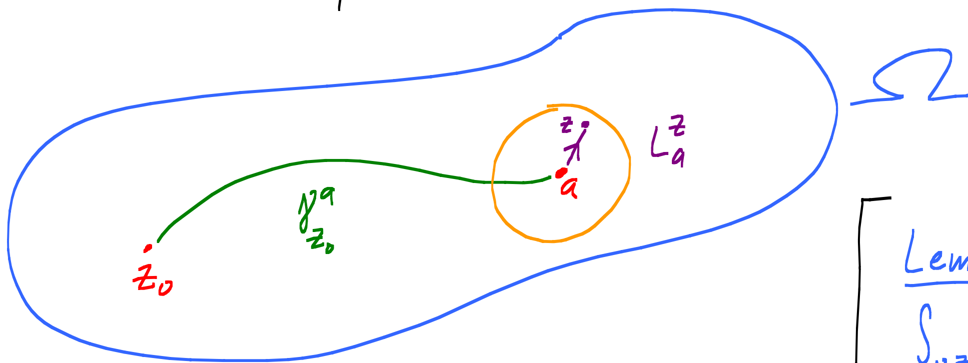
# Lesson 13 The Cauchy Theorem

Assuming $\int_\gamma f\,dz = 0$ for all closed $\gamma$ in a domain $\Omega$ ($f$ analytic)

we want to cook up $F$ with $F' = f$




$$F(z) = \int_{\gamma_{z_0}^z} f(w)\,dw$$

$$= \Big( \underbrace{\int_{\gamma_{z_0}^a}}_{\text{Constant}} + \int_{L_a^z} \Big) f\,dw$$

Lemma:

$$\int_{\gamma_A^z} c\,dw = c(z-A)$$

$$\parallel$$

$$\int_{\gamma_A^z} \frac{d}{dw}[cw]\,dw$$

$$= cz - cA \checkmark$$

Want: $F'(z) = \dfrac{d}{dz}\left( \int_{L_a^z} f\,dw \right)$

$f$ analytic $\Rightarrow$ $f$ continuous. So near $a$:

$$f(z) = f(a) + E(z) \quad \text{where } E(z) \to 0 \text{ as } z \to a.$$

$$DQ = \frac{F(z)-F(a)}{z-a} = \frac{\int_{L_a^z} f\,dw}{z-a}$$

$$= \frac{\int_{L_a^z} (f(a) + E(w))\,dw}{z-a}$$

$$= \underbrace{\frac{f(a)(z-a)}{z-a}}_{f(a)} + \underbrace{\frac{1}{z-a}\int_{L_a^z} E(w)\,dw}_{\mathcal{E}(z)}$$

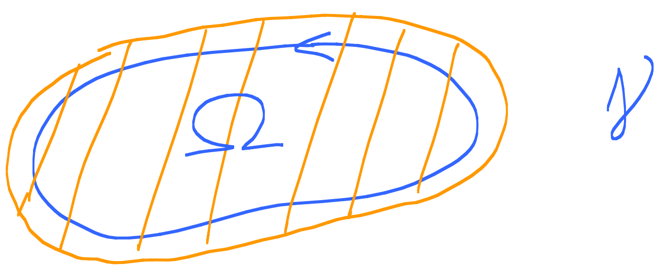
where $$|\mathcal{E}(z)| = \frac{1}{|z-a|}\left|\int_{L_a^z} E(w)\,dw\right|$$

$$\leq \frac{1}{|z-a|}\cdot\underbrace{\left(\operatorname*{Max}_{L_a^z}|E(w)|\right)}_{\to 0 \text{ as } z\to a}\underbrace{\text{Length}(L_a^z)}_{|z-a|}$$

So $\mathcal{E}(z) \to 0$ as $z \to a$ and we get $F'(a) = f(a)$.

Read 4.4 b, skip a.

Cauchy's Theorem: Suppose $f$ is "analytic inside and on" a simple closed curve $\gamma$. Then $\int_\gamma f\,dz = 0$.




f analytic on ▨

$\Omega$ = inside of $\gamma$

Green's Theorem: $$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Pf of Cauchy's Thm: $f(x+iy) = u(x,y) + i v(x,y)$

$z(t) = x(t) + i y(t)$

$$\int_\gamma f\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^b \left[u(x(t),y(t)) + i\,v(x(t),y(t))\right]\left(x'(t) + i y'(t)\right)dt$$

$$= \int_a^b \left(u\dot{x} \underset{i^2=-1}{-} v\dot{y}\right)dt + i\int\left(u\dot{y} + v\dot{x}\right)dt$$

$$\boxed{= \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx}$$

<u>Note:</u> $\int_a^b u(x(t), y(t))\,\underbrace{\frac{dx}{dt}\,dt}_{dx} = \int_\gamma u\,dx$

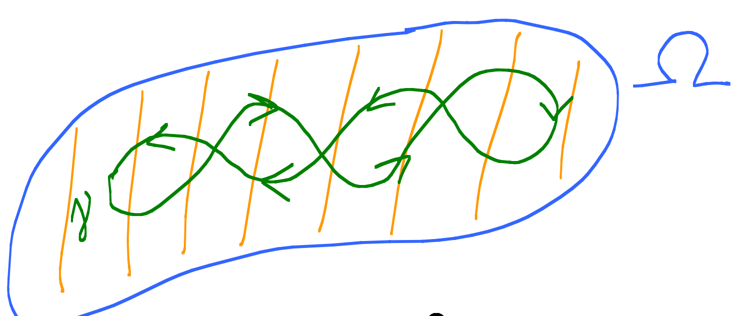
Green's

$$= \iint_\Omega \Big(\underbrace{\frac{\partial}{\partial x}(-v) - \frac{\partial u}{\partial y}}_{u_y = -v_x}\Big)\,dx\,dy + i\iint_\Omega \Big(\underbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}_{u_x = v_y}\Big)\,dx\,dy$$

$$= 0$$

<u>Cauchy's Thm #2:</u> Suppose $\Omega$ is a domain with no holes ($\Omega$ is "simply connected"), and $f$ analytic on $\Omega$. Then $\int_\gamma f\,dz = 0$ for every <u>closed</u> $\gamma$ in $\Omega$.

<u>Why:</u>



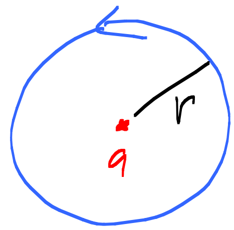

$$\int_\gamma f\,dz = \text{Sum of } \int_{\text{loops}} f\,dz = 0 \quad \text{by Cauchy's \#1.}$$

**Cor:** Using "Big fact" about getting $F$ with $F' = f$

$\iff \int_\gamma f\,dz = 0$ for closed $\gamma$, we see that

analytic fcns on a domain with no holes have analytic antiderivatives.

**Important calculation:**




$C_r(a)$ : $z(t) = a + re^{it}$
$0 \le t \le 2\pi$

$$\int_{C_r(a)} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{(a+re^{it})-a}\,\underbrace{ire^{it}}_{z'(t)}\,dt$$

$$= i\int_0^{2\pi} 1\,dt = 2\pi i$$

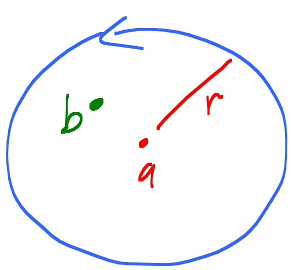
Consequence: Cauchy Thm fails on $\mathbb{C} - \{a\}$ ← domain with a hole

Also $\frac{1}{z-a}$ has no analytic antiderivate on $\mathbb{C} - \{a\}$.

$$\left[\ \int_{C_r(a)} F'\,dz = F(\text{END}) - F(\text{START}) = 0\ .\right]$$

Of course: $\frac{d}{dz}\log_\alpha(z-a) = \frac{1}{z-a}$ ← fails to hook up on $\mathbb{C} - \{a\}$.

**Big fact:**




$C_r(a)$

$$\int_{C_r(a)} \frac{1}{z-b}\,dz = \begin{cases} 2\pi i & b \text{ inside } C_r(a) \\ 0 & b \text{ outside} \end{cases}$$

↑ by Cauchy

why: Let $F(w) = \displaystyle\int_{C_r(a)} \frac{1}{z-w}\, dz$

$$DQ = \int DQ \longrightarrow \int_{C_r(a)} \frac{d}{dw}\left[\frac{1}{z-w}\right] dz$$

$$F'(w) = \int_{C_r(a)} \frac{1}{(z-w)^2}\, dz$$

$$= \int_{C_r(a)} \frac{d}{dz}\left[\frac{-1}{z-w}\right] dz$$

$$= 0$$

Aha! $F' \equiv 0$. So $F$ is constant.

Since $F(a) = 2\pi i$, the constant is $2\pi i$

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Remarks: Stokes' $\Longrightarrow$ Green's

$$\int_\gamma \omega = \iint_\Omega d\omega$$

$\omega = P\,dx + Q\,dy \leftarrow$ differential 1-form

$$d\omega = dP \wedge dx + dQ \wedge dy$$

$$= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy$$

$$= \frac{\partial P}{\partial x}\underbrace{dx \wedge dx}_{=0} + \frac{\partial P}{\partial y}\underbrace{dy \wedge dx}_{-dx \wedge dy} + \frac{\partial Q}{\partial x}dx \wedge dy + \frac{\partial Q}{\partial y}\underbrace{dy \wedge dy}_{=0}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underbrace{dx \wedge dy}_{\text{element of area } dA}$$