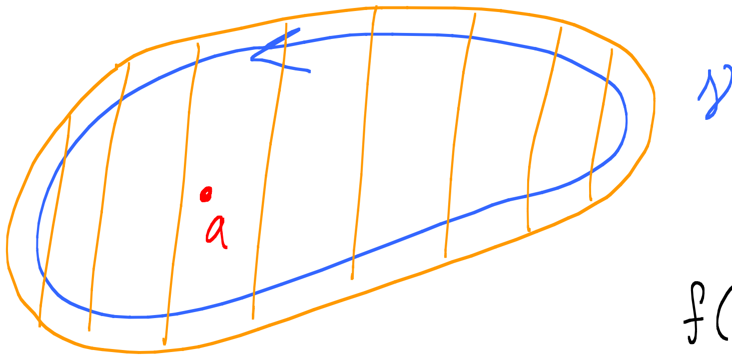
Lesson 15 Liouville's Theorem & the Fundamental Theorem of Algebra

12, 13, 14 due Wed., Exam 1 in class next Monday, Sept. 30, Lessons 1-16



f analytic on 

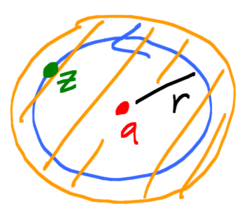
$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

Higher order Cauchy Integral Formulas: $\boxed{f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2}\, dz}$

$\vdots$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

The Cauchy Estimates:



$C_r(a)$ f analytic on 

$$|f^{(n)}(a)| \le \frac{n!\,M}{r^n} \quad \text{where } M = \max_{\text{on } C_r(a)} |f|.$$

Way false $\mathbb{R} \to \mathbb{R}$: $\sin(Nx) \leftarrow$ Bounded by 1, wiggles like crazy, N big

Why: $$|f^{(n)}(a)| = \frac{n!}{2\pi} \left| \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz \right| \le$$

$$\frac{n!}{2\pi} \left(\max_{z \in C_r(a)} \frac{|f(z)|}{|z-a|^{n+1}} \right) \left(\underbrace{\text{Length}(C_r(a))}_{2\pi r} \right) \qquad |z-a| = r$$

$$\leq \frac{n!\,M}{r^n} \quad \checkmark$$

Liouville's Theorem: A $\underbrace{\text{bounded}}_{|f(z)|\leq M}$ $\underbrace{\text{entire function}}_{\text{analytic on } \mathbb{C}}$ must be $\underbrace{\text{constant}}_{f(z)\equiv c}$.

Way false $\mathbb{R}\to\mathbb{R}$: $\sin x$

Pf: Pick any point $a\in\mathbb{C}$. First order Cauchy Estimate:

$$|f'(a)| \leq \frac{1!\,M}{r^1} \quad \text{where} \quad M = \max_{C_r(a)} |f|$$

$$\leq \frac{m}{r} \quad \text{if} \quad |f(z)|\leq m \text{ on } \mathbb{C}.$$

Aha! Can let $r\to\infty$. See $f'(a)=0$.

But a was arbitrary, So $f'\equiv 0$. So $f\equiv$ const.

Next: Liouville's $\Longrightarrow$ Fund. Thm. Alg.

Basic polynomial estimate: $P(z) = a_N z^N + a_{N-1}z^{N-1} + \cdots + a_1 z + a_0$ where $a_N \neq 0$ and $N\geq 1$. There are real constants $0<A<B$ and a radius $R>0$ such that

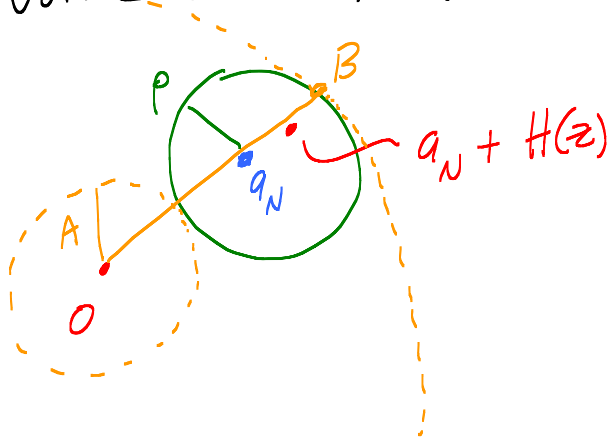
$$A|z|^N < |P(z)| < B|z|^N \quad \text{when } |z|>R.$$

Think: $|P(z)|$ blows up like $|z|^N$ as $|z|\to\infty$.

Consequence: All zeroes of $P(z)$ must fall in $\overline{D_R(0)}$. ← closure

Why: $P(z) = z^N \left[a_N + \underbrace{\frac{a_{N-1}}{z} + \frac{a_{N-2}}{z^2} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}}_{H(z)} \right]$

Notice that $H(z) \to 0$ as $|z| \to \infty$.



Choose $\rho > 0$ with $\rho < |a_N|$.

Since $H(z) \to 0$ as $|z| \to \infty$, there is a radius $R > 0$ such that $|H(z)| < \rho$ if $|z| > R$.

Let $A = |a_N| - \rho$
$B = |a_N| + \rho$

We have $A < |a_N + H(z)| < B$ when $|z| > R$.

Mult by $|z^N|$: $A|z|^N < |P(z)| < B|z|^N$ when $|z| > R$. ✓

Notice: By taking ρ small, can squeeze A and B close together $A < |a_N| < B$.

Fundamental Theorem of Algebra: A polynomial $P(z)$ of degree $N \geq 1$ has a root in $\mathbb{C}$. [$\exists\ z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.]

Why: Suppose P has no roots in $\mathbb{C}$. Then $f(z) = \frac{1}{P(z)}$ is an entire f'cn! Basic poly est: $A|z|^N < |P(z)| < B|z|^N$ when $|z| > R$.

$$|f(z)| = \frac{1}{|P(z)|} < \frac{1}{A|z|^N} \quad \text{when } |z| > R.$$

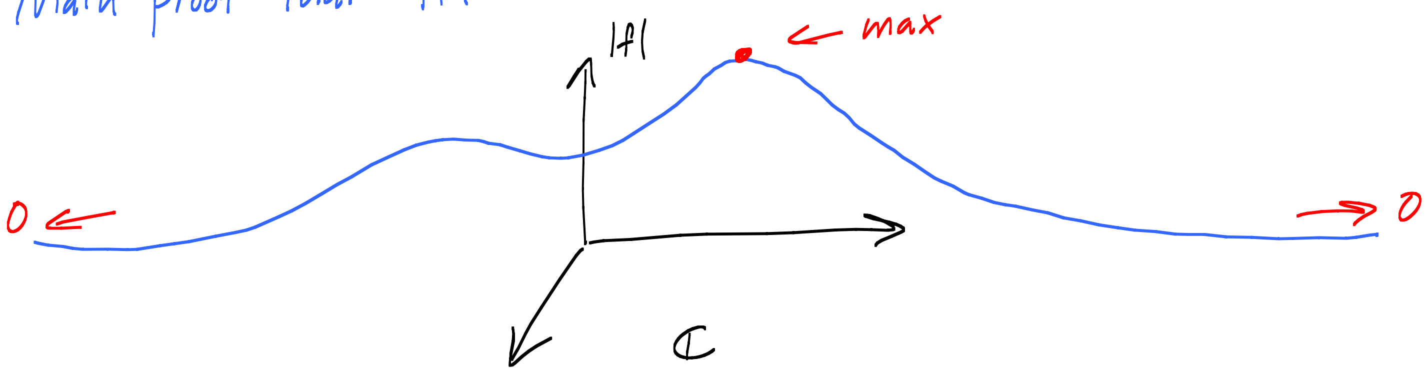
Aha! $|f(z)| \to 0$ as $|z| \to \infty$. So f is a bounded entire fcn. Liouville's $\Longrightarrow$ $f \equiv$ const.

So P is constant too. ↯

[P can't be constant: $\frac{d^N}{dz^N} P(z) = N!\, a_N \neq 0$.]

Conclude that P must have a root. ✓

Math proof that $|f|$ is bounded.



Know $|f(z)| \to 0$ as $|z| \to \infty$. So $\exists\, R_0 > 0$ such that $|f(z)| < 1$ if $|z| > R_0$.

f analytic $\Rightarrow$ f continuous. $\Longrightarrow$ $|f|$ continuous.

$|f|$ continuous on $\overline{D_{R_0}(0)} = \{|z| : |z| \le R_0\}$, a closed and bounded set. So $|f|$ assumes its max, say M, in $\overline{D_R(0)}$. Now $\max\{1, M\}$ is a bound for $|f|$ on $\mathbb{C}$.

<u>Remark</u>: Fund. Thm. Alg $\Longrightarrow$ Polys can be factored over $\mathbb{C}$.

Euclidean algorithm:

$$\begin{array}{r@{\,}l} & \;\; q(z) \\ P(z) & \big) \overline{\,Q(z)\qquad\quad} \\ & \quad \ddots \\ & \qquad \overline{\qquad\qquad} \\ & \qquad\quad r(z) \leftarrow \text{remainder} \end{array}$$

$$\begin{array}{r@{\,}l} & \;\; q(z) \\ (z-a) & \big) \overline{\,P(z)\qquad\quad} \\ & \quad \ddots \\ & \qquad b \in \mathbb{C} \end{array}$$

$$P(z) = \underbrace{q(z)(z-a) + b}_{\text{if } P(a)=0. \text{ see } b=0.}$$

(plug $z=a$ in)

$$P(z) = a_N \underbrace{(z-r_1)(z-r_2)\cdots(z-r_N)}_{\text{some } r_j\text{'s can be same}}$$

$$= a_N (z-r_1)^{m_1} \cdots (z-r_n)^{m_n}$$

$$m_1 + \cdots + m_n = N$$