Lesson 16 cont'd Harmonic fcns, Poisson's formula   Lessons 15, 16, 19 due Wed. 1

Nice facts: 1) If φ is a C^2-smooth harmonic fcn, then

$$f = \varphi_x - i\varphi_y \text{ is analytic.}$$

2) Max princ for analytic fcns $\Longrightarrow$ max princ for harmonic fcns. $|e^f| = e^{\operatorname{Re} f}$

EX using 2: Given harmonic fcn u on a simply connected domain Ω

Step 1: Cook up a harmonic conjugate for u on Ω.

Hmmm. $F = u_x - iu_y$ is analytic.

Whoa! If $f = u + iv$, then $\underset{\substack{\uparrow \\ F!}}{f'} = u_x + iv_x = u_x - iu_y$

Big idea: We know how to find an analytic antiderivative

$$f(z) = \int_{\gamma_a^z} F(w)\,dw.$$

Write $f(z) = U + iV$.

Then $f'(z) = F(z)$

$$U_x + i\underline{\underline{V_x}} = u_x - i\underline{\underline{u_y}}$$

$$\underline{\underline{V_y}} - iU_y = \underline{\underline{u_x}} - iu_y$$

See CR Eqns! V is a harmonic conjugate for U!

Next: $f(z) = u + iV$ is analytic.

Assume u has a local max at z_0.

Then, since $|e^f| = e^u$, we see that the modulus of e^f has a local max at z_0.

Max Princ $\Rightarrow$ $e^f \equiv$ const. $= K$

Aha! $$\underbrace{\frac{d}{dz}\left(e^{f(z)}\right)}_{f'(z)\,e^{f(z)}} = \frac{dK}{dz} \equiv 0$$

So $f'(z) \equiv 0$ and we see that

$f = u + iv$ is const on Ω. So $u \equiv$ const ✓

<u>Note</u>: Max prin for harmonic fcns $\Longrightarrow$

Min princ.

<u>Why</u>: Apply max princ to $-u$.

<u>Question</u>: Is there a minimum principle for analytic fcns?

No! z^2, for example on $D_1(0)$. Modulus $= 0$ at 0, a min, but $z^2 \not\equiv 0$.

<u>But</u>, if f is <u>non-vanishing</u>, the max princ for $1/f$ $\Longrightarrow$ min princ for f.

Liouville's Theorem for harmonic fcns: If u is harmonic on $\mathbb{C}$ and bounded from above (or below), then $u \equiv$ const.

Why: First, let v be a harm conj for u on $\mathbb{C}$.

Write $f = u + iv$. Say $u \le M$ on $\mathbb{C}$.

$$|e^f| = |e^{u+iv}| = \underbrace{|e^u|}_{e^u}\ \underbrace{|e^{iv}|}_{1} = e^u \le e^M \text{ on } \mathbb{C}$$

f analytic $\Rightarrow e^f$ analytic (entire!). e^f bdd entire. Liouville's $\Rightarrow e^f \equiv K$, a const.

$\frac{d}{dz}(e^f) = f' e^f \equiv 0$ shows $f' \equiv 0$.

So f const. So $u \equiv$ const. ✓

$f = u + iv$

Alternatively, (#2 on Exam 1) $|e^f| = e^u$. If $e^f \equiv K$, a const, then $|K| \equiv e^u$. So $u \equiv \ln|K|$, a const.

Aha! Can use C-R Eqns

$$\begin{cases} v_x = -u_y \equiv 0 \\ v_y = u_x \equiv 0 \end{cases}$$

$\nabla v \equiv 0$. So v const too. So $f = u + iv$ is const.

<u>Famous problem</u>: Given a continuous real valued fcn φ on the unit circle $\left[\varphi(\theta) : \begin{array}{l} 0 \le \theta \le 2\pi \\ \varphi(0) = \varphi(2\pi) \end{array} \right]$

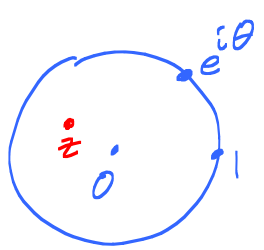
<u>Dirichlet problem</u>: Find a real valued fcn u on $\overline{D_1(0)}$ that is <u>continuous</u> on $\overline{D_1(0)}$ and <u>harmonic</u> on $D_1(0)$ such that

$$u(e^{i\theta}) = \varphi(\theta) \quad \text{for } 0 \le \theta \le 2\pi.$$

<u>Note</u>: u solves steady state heat prob. $u = \varphi$ on circle.

<u>Poisson's formula</u>:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} u(e^{i\theta})\, d\theta$$



True for u harmonic on a disc $D_R(0)$ with $R > 1$.

<u>Why</u>: Cook up a harmonic conjugate v for u on $D_R(0)$.

Play around with the Cauchy integral:

$$f = u + iv$$

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{1}{z-w} + \text{Something}\right] f(w)\,dw$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\underbrace{\left[\left(\frac{1}{z-e^{i\theta}} + \text{Something}\right) ie^{i\theta}\right]}_{\text{Poisson: There is "something" that makes this real valued without changing the }\int.}f(e^{i\theta})\,d\theta$$

Get "Poisson kernel" $P(z,\theta)$

$$\underbrace{f(z)}_{u+iv} = \frac{1}{2\pi}\int_0^{2\pi}\underbrace{P(z,\theta)}_{\text{real!}}\ \underbrace{f(e^{i\theta})}_{u+iv}\,d\theta$$

Take real part. Get Poisson formula for u.

Big wow! Poisson's formula solves the Dirichlet problem! (proof later)

Remark: Max principle shows sol^n is unique.

why: u_1, u_2 two sol^{n}s. Then

$u_1 - u_2$ is harmonic on $D_1(0)$ and zero on $C_1(0)$.

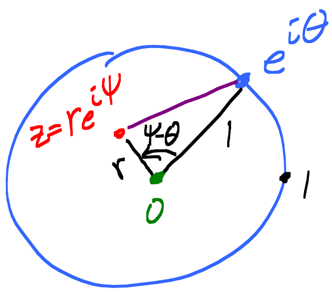
Max princ $\Longrightarrow$ $u_1 - u_2 \le 0$ on $\overline{D_1(0)}$.

Repeat for $u_2 - u_1$. Get $u_2 - u_1 \le 0$.

So $|u_1 - u_2| \equiv 0$ and $u_1 \equiv u_2$.

Other forms of Poisson's formula:

$$u(re^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{\underbrace{1+r^2-2r\cos(\theta-\psi)}_{|re^{i\psi}-e^{i\theta}|^2}}\,\varphi(\theta)\,d\theta$$



See book for $D_R(0)$ formula.

Ave prop: Analytic: $f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \underbrace{\frac{f(a+re^{i\theta})}{(a+re^{i\theta})-a}\, ire^{i\theta}}_{i f(a+re^{i\theta})}\,d\theta$

a center

Take real part for harm fcn ave prop.