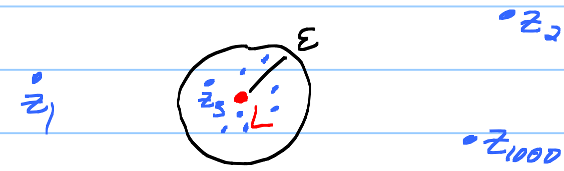
Lesson 19 Power series

Read 5.1 and 5.2 carefully.

Great fact: $\mathbb{R}\to\mathbb{R}$ proofs about limits, seq, series go over to $\mathbb{C}\to\mathbb{C}$ line by line.

$$\lim_{n\to\infty} z_n = L$$



$$|z_n - L| < \varepsilon \text{ for } n > N = 1000$$

Typical fact: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$ if limits exist and $b_n \to L \neq 0$.

Power series: $\displaystyle f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ centered at a.

EX: Geometric series:

$$1 + z + z^2 + z^3 + \cdots + z^N = S_N$$

$$- \quad z + z^2 + z^3 + \cdots + z^N + z^{N+1} = z S_N$$

$$1 - z^{N+1} = (1-z) S_N$$

N-th Partial sum: $\displaystyle S_N = \frac{1-z^{N+1}}{1-z} = \underbrace{\frac{1}{1-z}}_{\text{limit!}} + \underbrace{\frac{-z^{N+1}}{1-z}}_{\varepsilon_N(z)}$

Big fact: $S_N(z) \to \frac{1}{1-z}$ as $N\to\infty$ when $|z|<1$.

And the convergence is uniform on $D_r(0)$ when $0<r<1$.

Why: Denom. estimate: $|1-z| \geq \big|1-|z|\big| = \pm(1-|z|)$

$$= \begin{cases} 1-|z| & \text{if } |z|<1 \\ |z|-1 & \text{if } |z|>1 \end{cases}$$

So
$$|\mathcal{E}_N(z)| = \left|\frac{-z^{N+1}}{1-z}\right| \le \underbrace{\frac{|z|^{N+1}}{1-|z|}}_{\substack{\to 0 \\ \text{as } N\to\infty \\ \text{if } |z|<1.}} \le \underbrace{\frac{r^{N+1}}{1-r}}_{\substack{\to 0 \\ \text{as } N\to\infty \\ \text{if } |z|<r<1.}}$$

Get same error estimate $|\mathcal{E}_N(z)|<\varepsilon$ for <u>all</u> z in $D_r(0)$

uniformly

<u>Fact</u>: $S_N(z)$ <u>diverges</u> when $|z|\ge 1$ because the terms $|z^n|=|z|^n$ do not go to zero.

<u>Thm</u>: If $\sum_{n=1}^{\infty} z_n$ converges then $z_n \to 0$ as $n\to\infty$.

<u>Note</u>: This is one-way implication $\Longrightarrow$. The reverse can fail. <u>Ex</u>: $\sum_{1}^{\infty} \frac{1}{n} = \infty$

<u>Pf of thm</u>: $L = \sum_{n=1}^{\infty} z_n$ $\qquad S_N = \sum_{1}^{N} z_n$

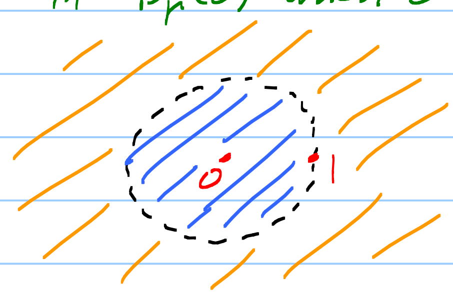
$$z_N = S_N - S_{N-1}$$
$$\downarrow \qquad \downarrow$$
$$L - L = 0 \quad \text{as } N\to\infty. \checkmark$$

<u>Geometric series</u>: $\sum_{0}^{\infty} z^n$ converges on $D_1(0)$ to $\frac{1}{1-z}$

It converges <u>uniformly</u> in $D_r(0)$ when $0<r<1$.

It diverges if $|z|\ge 1$.

Div $\qquad$ Conv



Radius of convergence $= R = 1$.

<u>Def^n</u>: A seq of functions $f_n(z)$ on a domain Ω is said to <u>converge uniformly on compact subsets</u> to $f(z)$ if, for each compact $K \subset \Omega$ ["compact" means "closed and bounded"] and $\varepsilon > 0$, there is an N such that $|f_n(z) - f(z)| < \varepsilon$ for $\underline{\underline{\text{all}}}$ z in K. Notation: $f_n \twoheadrightarrow f$

<u>Remark</u>: When $\Omega = D_1(0)$, unif conv on compacts $\Longleftrightarrow$ unif conv on $D_r(0)$ for every r with $0 < r < 1$.

$|z|$ continuous on compact K, so it assumes it max ρ $\underline{\underline{\text{in}}}$ K. $\rho < 1$

Take r with $\rho < r < 1$.

<u>Fantastic complex variable fact</u>: Morera's thm $\Longrightarrow$

If a seq of analytic fcns f_n converges uniformly on compact subsets to f, the f is analytic!

<u>Why</u>: Can restrict to a disc. f_n analytic $\Longrightarrow$

$\int_\gamma f_n \, dz = 0$ for closed loops in disc.

$tr(\gamma)$ is closed, bdd so compact.

$$\left| \int_\gamma f \, dz - \underbrace{\int_\gamma f_n \, dz}_{=0} \right| = \left| \int_\gamma (f - f_n) \, dz \right| \le \underbrace{\left(\max_\gamma |f - f_n|\right)}_{\to 0} \text{Length}(\gamma)$$

Conclude $\int_\gamma f \, dz = 0$. Morera's $\Longrightarrow$ f analytic!

Important fact: Uniform limits of continuous fcns are continuous.

(Needed to limit fcn f is cont. to use Morera's.)