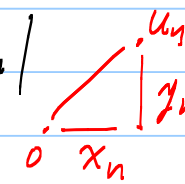
Lesson 20 Analytic functions and power series 5.2

HWK 6 due Friday

Def^n: $\sum_{n=1}^{\infty} u_n$ converges absolutely if $\sum_{n=1}^{\infty} |u_n| < \infty$.

Fact: Absolutely convergent series converge!

Why: Write $u_n = x_n + i y_n$. Then $|x_n| \le |u_n|$, $|y_n| \le |u_n|$



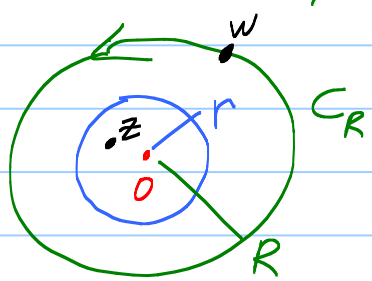
So $\sum |u_n| < \infty \Rightarrow \sum_1^{\infty} |x_n|$ and $\sum_1^{\infty} |y_n|$ finite.

$\mathbb{R}$ case yields that $\sum x_n$ & $\sum y_n$ converge.

Now $\sum_1^N u_n = \left(\sum_1^N x_n\right) + i\left(\sum_1^N y_n\right)$ converges. ✓

This is a deep fact related to completeness (every Cauchy seq has a limit). $\mathbb{R}$ is complete. So is $\mathbb{C}$!

Assume f is analytic on a disc bigger than $D_R(0)$.



$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w-z}\,dw$$

$$\left(\left|\frac{z}{w}\right| < \frac{r}{R} < 1\right) \qquad = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\,\frac{1}{1-\frac{z}{w}}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\left[1 + \left(\frac{z}{w}\right) + \left(\frac{z}{w}\right)^2 + \cdots + \left(\frac{z}{w}\right)^N + \underbrace{E_N\left(\frac{z}{w}\right)}_{\frac{\left(\frac{z}{w}\right)^{N+1}}{1-\frac{z}{w}}}\right] dw$$

$$= \sum_{0}^{N} \underbrace{\left(\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w^{n+1}}\,dw\right)}_{\frac{f^{(n)}(0)}{n!}\ !} z^n + \underbrace{\mathcal{E}_N(z)}_{\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w}\,\frac{\left(\frac{z}{w}\right)^{N+1}}{1-\frac{z}{w}}\,dw}$$

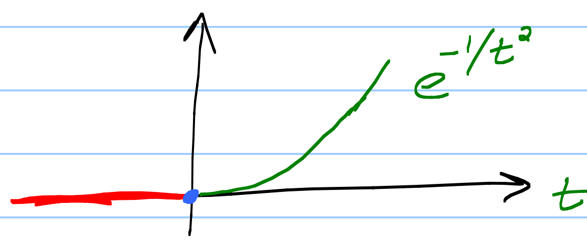
where $\displaystyle |\mathcal{E}_N(z)| \le \frac{1}{2\pi}\left(\underbrace{\max_{C_R}|f|}_{M}\cdot\frac{1}{R}\cdot\frac{\left(\frac{r}{R}\right)^{N+1}}{1-\frac{r}{R}}\right)\underbrace{\text{Length}(C_R)}_{2\pi R}$

$$\le \frac{M\left(\frac{r}{R}\right)^{N+1}}{1-\frac{r}{R}} \longrightarrow 0 \quad \text{as } N\to\infty.$$

Conclude $\displaystyle\sum_{n=0}^{\infty} a_n z^n \longrightarrow f(z)$ uniformly on $D_r(0)$.

(And absolutely.) where $\displaystyle a_n = \frac{f^{(n)}(0)}{n!}$.

Way false $\mathbb{R}\to\mathbb{R}$:



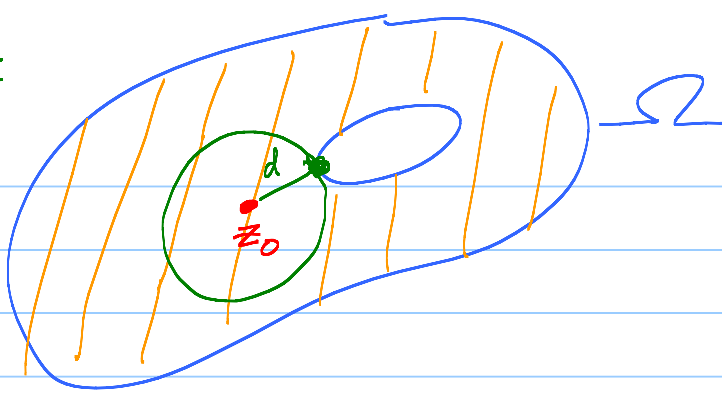
is C^∞-smooth. Taylor series at $t=0$ is $\equiv 0$ series $\neq f(t)$.

EX: $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ converges absolutely

and uniformly on every $D_r(0)$ for any $r>0$.

Say Radius of convergence (RoC) is ∞.

Theorem:

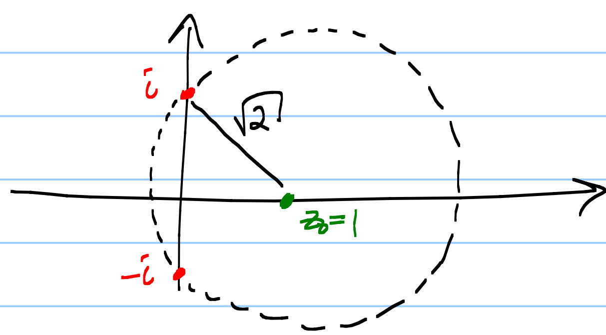


f analytic on a domain Ω. $z_0 \in \Omega$.

$f(z) =$ Convergent power series about z_0 with RoC at least as big as $d = \operatorname{dist}(z_0, \underbrace{\partial\Omega}_{\text{the boundary of } \Omega})$.

$d = \sup\{r : D_r(z_0) \subset \Omega\}$

EX: What is the RoC about $z_0 = 1$ of the power series for $f(z) = \frac{1}{z^2+1}$?



Thm $\Rightarrow$ RoC $\geq \sqrt{2}$.

But, if RoC $> \sqrt{2}$, the series converges to an analytic fcn perfectly well behaved at $\pm i$, $f \equiv$ power series on $D_{\sqrt{2}}(1)$ and it blows up at $\pm i$. $\lightning$.

So RoC $\leq \sqrt{2}$. Must $= \sqrt{2}$.

Famous series: e^z, $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} -$

$\sinh z = \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

all have RoC $= \infty$

Geometric series: $1+z+z^2+\cdots = \frac{1}{1-z}$ RoC $=1$

Ratio Test: $\sum_{n=0}^{\infty} u_n$.

Suppose $\left|\frac{u_{n+1}}{u_n}\right| \longrightarrow L$ as $n\to\infty$.

$L<1$ series converges absolutely.

$L>1$ series diverges

$L=1$ test fails.

Why: Compare tail end of series with a geometric series.

EX: What is the radius of conv of

$$\sum_{n=0}^{\infty} \underbrace{n3^n z^n}_{u_n}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)3^{n+1}z^{n+1}}{n\ 3^n\ z^n}\right| = 3\frac{n+1}{n}|z| \longrightarrow 3|z| \begin{cases} <1 & \text{conv} \\ >1 & \text{div} \end{cases}$$

So conv: $|z|<\frac{1}{3}$

div: $|z|>\frac{1}{3}$

So RoC $=\frac{1}{3}$.

Be carefull: Ratio test doesn't always work.

EX: Let a_n be an enumeration of all the points in the unit circle with rational real and imaginary parts.

$f(z)=\sum_{n=0}^{\infty} a_n z^n$ is analytic on $D_1(0)$. RoC $=1$.

Comparison test: Given $M_j>0$. Suppose $\sum_{j=1}^{\infty} M_j<\infty$.

If $z_j \in \mathbb{C}$ and $|z_j| < M_j$, then $\sum_{j=1}^{\infty} z_j$ Converges.

EX: $f(z)$: $|a_n z^n| = \underbrace{|a_n|}_{<1} |z|^n < |z|^n$ ← Aha! Compare series to a geom series when $|z|<1$.

So $RoC \geq 1$.

Hadamard's Formula: $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. $R = RoC$

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

where $\limsup_{n \to \infty} r_n = \lim_{N \to \infty} \sup \{ r_n : n \geq N \}$