# Lesson 22 Hadamard's formula, differentiating power series

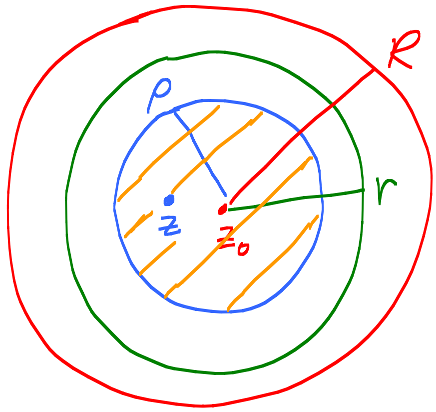
Hwk 7 due Wed



$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{N\to\infty} \underbrace{\sup\{\sqrt[n]{|a_n|} : n \geq N\}}$$

non-increasing seq in $N$ bounded below by $0$ has a limit $\geq 0$.

$R = 0, \infty$ ok




Sup = least upper bound

feeling: like a max, but might not be in set

Think: Those Sup's are sliding down to $\frac{1}{R}$.

<u>Why Hadamard's works:</u>

Take $z_0 = 0$.
Assume $R \neq 0, \neq \infty$.

$$\frac{1}{R} < \frac{1}{r} < \frac{1}{\rho}$$

$\sup\{\sqrt[n]{|a_n|} : n \geq N\}$ sliding down to $\frac{1}{R}$

So eventually, there is an $N$ such that

$$\sup\{\sqrt[n]{|a_n|} : n \geq N\} < \frac{1}{r}$$

$\frac{1}{r}$ is an upper bound for the set

$$\sqrt[n]{|a_n|} < \frac{1}{r} \quad \text{for } n \geq N$$

$$|a_n| < \frac{1}{r^n} \quad \text{for } n \geq N$$

So, if $|z| < \rho$, then

$$|a_n z^n| = |a_n||z|^n < \underbrace{|a_n|}_{<\frac{1}{r^n}}\rho^n < \left(\frac{\rho}{r}\right)^n, \; n \geq N$$

Aha! Tail end of $\sum_0^\infty a_n z^n$ compares to conv geom series $\left(\frac{\rho}{r} < 1\right)$

in $D_\rho(0)$. Comparison test also yields <u>uniform</u> convergence.

Can make $\rho$ squeeze out as close as I want to $R$. So series converges on $D_R(0)$. So RoC is $\geq R$.

<u>Last step</u>: Show $\sum a_n z^n$ <u>diverges</u> when $|z| > R$. So RoC $= R$. Easy: If $|z| > R$:

$$\frac{1}{|z|} < \frac{1}{R} \leq \sup\{\sqrt[n]{|a_n|} : n \geq N\} \text{ for each } N.$$

So, for each $N$, there exists an $n \geq N$ such that

$$\frac{1}{|z|} < \sqrt[n]{|a_n|}$$

$$\frac{1}{|z|^n} < |a_n|$$

$$1 < |a_n z^n|$$

Aha! The terms don't $\to 0$ in $\sum a_n z^n$.

Series diverges. RoC $= R$.

<u>EX</u>: $\sum_0^\infty z^{n!} = z + z^2 + z^6 + z^{24} + \cdots$

$$a_n = \begin{cases} 1 & \text{if } n = m! \text{ for some } m \\ 0 & \text{otherwise} \end{cases}$$

$$\sup\{\sqrt[n]{|a_n|} : n \geq N\} = 1 \quad \text{for all } N.$$

LimSup $= 1$. So $\frac{1}{R} = 1$. $\boxed{R = 1}$

**Huge fact:** Can differentiate power series term by term.

$$f(z) = \sum_0^\infty a_n (z-z_0)^n \quad \leftarrow RoC = R$$

$$f'(z) = \sum_1^\infty n\, a_n (z-z_0)^{n-1} = \sum_0^\infty \underbrace{(n+1) a_{n+1}}_{= b_n \text{ below}} (z-z_0)^n$$

also $RoC = R$ !

**Why:** Know power series converge uniformly on $D_r(z_0)$, $r < R$.
Morera's $\Rightarrow$ limit is analytic. Write $f(z) = \sum_0^\infty a_n (z-z_0)^n$.
$f$ analytic $\Rightarrow$ $f'$ analytic on $D_R(z_0)$ too.

So it has a taylor series:

$$f'(z) = \sum_{n=0}^\infty b_n (z-z_0)^n \quad \text{where}$$

$$b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!} \cdot \frac{n+1}{n+1}$$

$$= (n+1) \frac{f^{(n+1)}(z_0)}{(n+1)!} = (n+1)\, a_{n+1} \checkmark$$

$$b_n z^n = \frac{d}{dz}\left[ a_{n+1} z^{n+1} \right]$$

**Note:** What we've done shows that RoC for $f'$ series is
$\geq$ RoC for $f$ series.
But it can't be bigger. (Could antidiff and see $f$ analytic

on a bigger disc with bigge RoC. ↯ )

Fun thing: Compute $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$1 + z + z^2 + \cdots = \frac{1}{1-z} \quad \text{if } |z| < 1 \leftarrow R$$

diff'tiate $$1 + 2z + 3z^2 + 4z^3 + \cdots = \frac{1}{(1-z)^2} \quad \text{if } |z| < 1$$

multiply by $z$ $$z + 2z^2 + 3z^3 + 4z^4 + \cdots = \frac{z}{(1-z)^2} \quad (*)$$

Aha! $z = \frac{1}{2}$: $$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-\frac{1}{2})^2} = 2$$

Similarly: diff $(*)$: $1 + 2^2 z + 3^2 z^2 + \cdots$ = formula

mult by $z$: $z + 2^2 z^2 + 3^2 z^3 + \cdots$ = new formula

Set $z = \frac{1}{2}$: Get $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Important fact: $$(z-z_0)^N \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n} = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^{N+n}}$$

Same RoC!

Why: Think of $z$ as fixed. Left hand side converges (or diverges) exactly when the right hand side does.

Properties of RoC mean they have the same RoC.

## Can solve ODEs with power series

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$$

Let $x$ go complex!

$$f''(z) - z f'(z) - f(z) = 0$$

Assume $f$ is analytic. See what happens.

$$f(z) = \sum_0^\infty a_n z^n$$

$$f'(z) = \sum_0^\infty (n+1) a_{n+1} z^n \leftarrow = \sum_1^\infty n a_n z^{n-1}$$

$$f''(z) = \sum_0^\infty (n+2)(n+1) a_{n+2} z^n$$

$$z f'(z) = \sum_1^\infty n a_n z^n$$

Plug into ODE:

$$\sum_0^\infty (n+2)(n+1) a_{n+2} z^n - \sum_1^\infty n a_n z^n - \sum_0^\infty a_n z^n \overset{\text{want}}{=} 0$$

$$\underbrace{\left[2\cdot 1\, a_2 - a_0\right]}_{=0} + \sum_1^\infty \underbrace{\left[(n+2)(n+1) a_{n+2} - (n+1) a_n\right]}_{\text{need } =0} z^n \overset{\text{want}}{=} 0$$

$$a_2 = \frac{1}{2} a_0 \leftarrow f(0) = a_0 \text{ (initial condition)}$$

$$\boxed{a_{n+2} = \frac{a_n}{n+2}} \quad \text{for } n \geq 1.$$

$\leftarrow$ Recursion formula.

Spit out one-by-one.

$a_2 = \frac{1}{2} a_0$ $\qquad$ $a_1 = f'(0)$

$a_4 = \frac{1}{4 \cdot 2} a_0$ $\qquad$ $a_3 = \frac{1}{3} a_1$

$a_6 = \frac{1}{6 \cdot 4 \cdot 2} a_0$ $\qquad$ $a_5 = \frac{1}{5 \cdot 3} a_1$

$\vdots$ $\qquad$ $\vdots$

$$f(x) = a_0 \Bigg( \qquad \qquad \Bigg) + a_1 \Bigg( \qquad \qquad \Bigg)$$