Lesson 23 Zeroes of analytic functions

$$f''(z) - z f'(z) - f(z) = 0$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad -\left(f(z) = \sum_{n=0}^{\infty} a_n z^n \right)$$

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad -\left(z f'(z) = \sum_{n=1}^{\infty} n a_n z^n \right)$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \qquad + \quad f''(z) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

$$0 \overset{\text{want}}{=} \underbrace{(2\cdot 1\, a_2 - a_0)}_{\text{straggler term } = 0} + \sum_{n=1}^{\infty} \Big[\underbrace{(n+2)(n+1) a_{n+2} - n a_n - a_n}_{=0} \Big] z^n$$

$$\text{IC} \begin{cases} a_0 = f(0) \\ a_1 = f'(0) \end{cases}$$

$$a_2 = \tfrac{1}{2} a_0$$

$$\boxed{a_{n+2} = \frac{a_n}{(n+2)}} \quad n \geq 1$$

Recursion formula

$n=1$: $a_3 = \dfrac{a_1}{3}$

$n=2$: $a_4 = \dfrac{a_2}{4} = \dfrac{a_0}{4\cdot 2}$

$n=3$: $a_5 = \dfrac{a_3}{5} = \dfrac{a_1}{5\cdot 3}$

$\vdots$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = a_0 + a_1 z + \tfrac{1}{2} a_0 z^2 + \tfrac{1}{3} a_1 z^3 + \cdots$$

$$= a_0 \Big(\underbrace{1 + \tfrac{1}{2} z^2 + \tfrac{1}{4\cdot 2} z^4 + \tfrac{1}{6\cdot 4\cdot 2} z^6 + \cdots}_{\cdots + \frac{z^{2n}}{2^n n!} + \cdots} \Big) + a_1 \Big(\underline{z + \tfrac{1}{3} z^3 + \tfrac{1}{5\cdot 3} z^5 + \tfrac{1}{7\cdot 5\cdot 3} z^7 + \cdots} \Big)$$

2 lin. ind. sol^{n}s y_1, y_2

Remark: I hate the formula $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

Better: Ratio test: $\lim_{\text{way out}\to\infty} \left|\frac{\text{Term way out there}}{\text{Term right before}}\right|$

$$= \lim_{n\to\infty} \left|\frac{a_{n+2} z^{n+2}}{a_n z^n}\right|$$

Aha! Recursion says $\frac{a_{n+2}}{a_n} = \frac{1}{n+2}$

$$= \lim_{n\to\infty} \frac{1}{n+2} |z|^2 = 0 < 1$$ ← Converges for all z,

$R = \infty$.

Good news! Sol^n is valid.
Let $z = x \in \mathbb{R}$. Get real C^∞ smooth gen^l sol^n. $a_0 = c_1$, $a_1 = c_2$

Hated formula: $\sum_{n=0}^{\infty} \frac{z^{2n}}{2^n n!}$ Let $w = z^2$

Use formula to get RoC R of $\sum \frac{w^n}{2^n n!}$.

$|w| < R$

$|z^2| < R$ RoC $= \sqrt{R}$

Philosophy: Analytic fcns are like polynomials (of deg ∞).

Zeroes of analytic fcns. $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $R > 0$

Suppose $f(0)=0$. Then $a_0 = \frac{f(0)}{0!} = 0$.

Case 1: All a_n's $=0$. Then $f(z)\equiv 0$.

Case 2: Not all a_n's $=0$. So there is a first nonzero a_n. Call it $a_N \neq 0$. $[a_n = 0$ for $n=0,1,2,\ldots,N-1]$

$$f(z) = a_N z^N + a_{N+1} z^{N+1} + \cdots \quad \leftarrow \text{RoC} = R$$

$$= z^N \Big(\underbrace{a_N + a_{N+1} z + a_{N+2} z^2 + \cdots}_{F(z)} \Big)$$

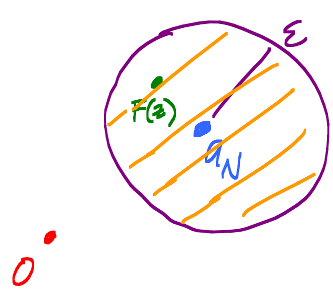
series converges for exactly same z's as one above!

has same RoC R!

So $f(z) = z^N F(z)$ where F is analytic on $D_R(0)$ and $F(0) = a_N \neq 0$.

Aha! The zero of f at $z=0$ is isolated.

Note: F analytic $\Rightarrow$ F continuous. $F(0) = a_N \neq 0$



Take $\varepsilon < |a_N|$. (Books: $\varepsilon = \frac{|a_N|}{2}$)

Def^n of continuity: $\exists\ \delta > 0$ such that $|F(z) - a_N| < \varepsilon$ when $|z-0| < \delta$.

F non-zero in $D_\delta(0)$.

$z=0$ is only zero of f in $D_\delta(0)$. ← Def^n of "isolated zero".

<u>Def^n</u>: N is the <u>order</u> (or multiplicity) of the zero at 0.

<u>Notice</u>:
$$0 = f(0) = a_0$$
$$0 = f'(0) = a_1$$
$$\vdots$$
$$0 = \frac{f^{(N-1)}(0)}{(N-1)!} = a_{N-1}$$

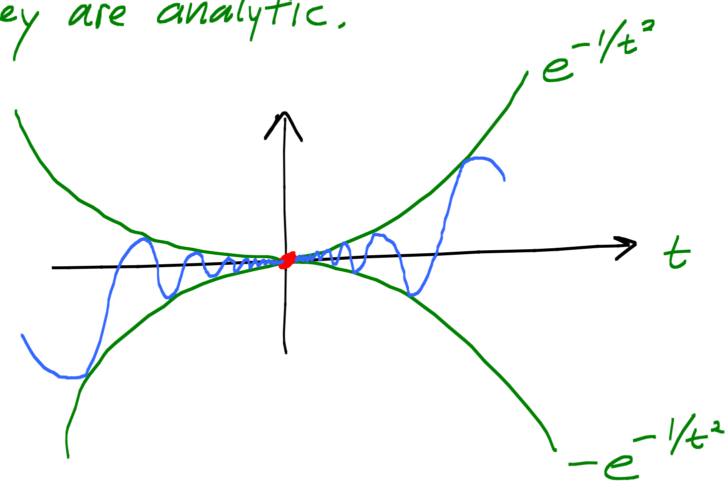
$$a_N = \frac{f^{(N)}(0)}{N!} \neq 0$$

order of first non-zero deriv at $z=0$.

<u>Think</u>: Zeroes of analytic fcns cannot "pile up" inside domain where they are analytic.

<u>Way false $\mathbb{R} \to \mathbb{R}$</u>:

$$h(t) = \begin{cases} e^{-1/t^2} \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$



h is C^∞ smooth on $\mathbb{R}$. Zeroes pile up at 0.

[∃ seq of zeroes → 0]

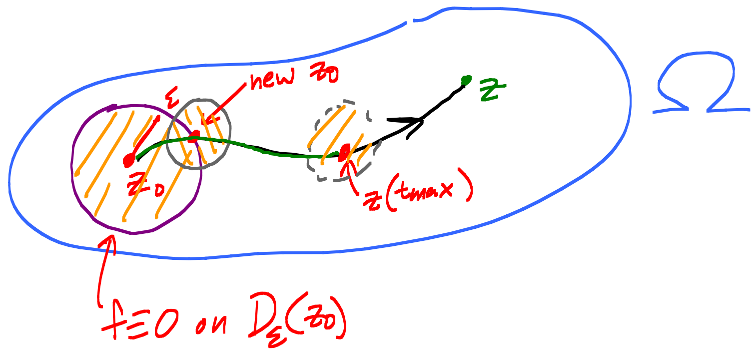
<u>Def^n</u>: $z_0 \in \Omega$ is an <u>accumulation point</u> of a set $S \subset \Omega$ if there is a seq $\{z_n\}_{n=1}^\infty$ in S with $z_n \neq z_0$ for each n such that $\lim_{n\to\infty} z_n = z_0$.

Same as saying: For every $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subset \Omega$,
$$\left(D_\varepsilon(z_0) - \{z_0\}\right) \cap S \neq \emptyset.$$

Def^n: "limit point" = "accum pt".

Big fact: (Identity theorem) Given an analytic fcn f on a domain Ω, if $Z_f = \{z \in \Omega : f(z) = 0\}$ has an accumulation pt at a point z_0 in Ω, then $f \equiv 0$ on Ω.

Why:



z_0 = lim pt of Z_f

Aha! Can keep extending zeroness along curve.

Math way: Let $t_{max} = \sup\{\tilde{\tau} : f(z(t)) = 0 \text{ for all } t < \tilde{\tau}\}$

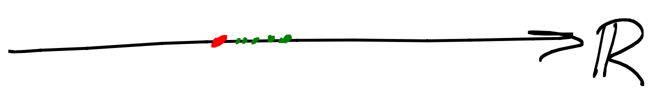
Aha! See $z(t_{max})$ is a lim. pt. of zeroes!

t_{max} not the max if $t_{max} < b$ ← end.

So $f = 0$ all along curve to z. $f(z) = 0$. ✓

Big consequence: e^z is the unique analytic fcn that extends e^x to $\mathbb{C}$.

Say F is another $\underbrace{F(z) - e^z}_{\equiv 0 \text{ on } \mathbb{C}} \equiv 0$ along $\mathbb{R}$.



(Every pt on $\mathbb{R}$ is a lim pt of $\mathbb{R}$ in $\mathbb{C}$.)