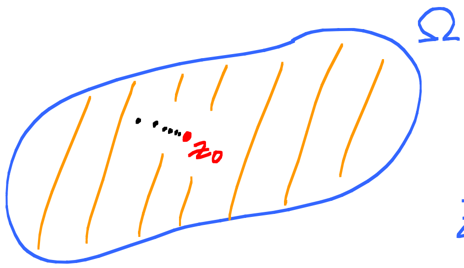
Lesson 24 Laurent expansions, isolated singularities

22, 23, 24 due Wed



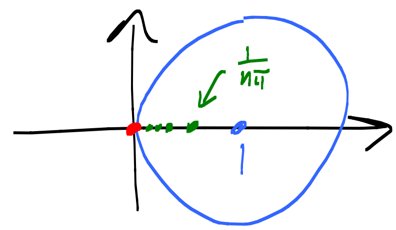
Identity Theorem (or Uniqueness Theorem)

f analytic on Ω

Z_f = zero set of f

If Z_f has a limit point $z_0 \in \Omega$, then $f \equiv 0$

EX: $f(z) = \sin \frac{1}{z}$ on $D_1(1)$



OK for zeroes to pile up at boundary pts

Cor: If an analytic fcn f on Ω vanishes on $D_\varepsilon(z_0) \subset \Omega$, then $f \equiv 0$ on Ω.

Cor: f, g analytic on Ω. If they agree on a set $S \subset \Omega$ with a limit pt in Ω, then $f \equiv g$ on Ω.

Note: Continuity $\Rightarrow z_0 \in Z_f$.

Application: All trig identities hold for "complex angles"!

Why: $\sin 2\theta = 2 \sin\theta \cos z$

$$\underbrace{\sin 2z}_{f(z)} \qquad \underbrace{2 \sin z \cos z}_{g(z)}$$

$\leftarrow$ f, g agree along $\mathbb{R}$ in $\mathbb{C}$.

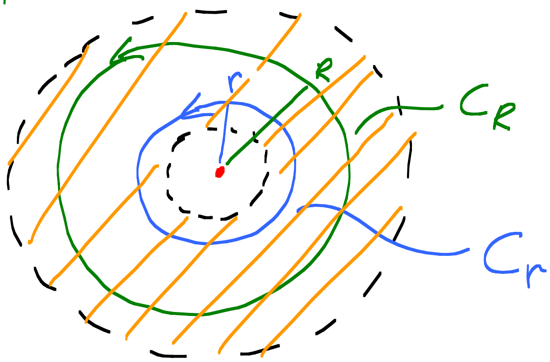
Every pt in $\mathbb{R}$ is a limit pt of $\mathbb{R}$ in $\mathbb{C}$. So $f \equiv g$ on $\mathbb{C}$.

EX: $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$ z z z

Step 1: Let β go complex: z. True for z.

Step 2: Then let $\alpha = w$ with z fixed. Repeat.

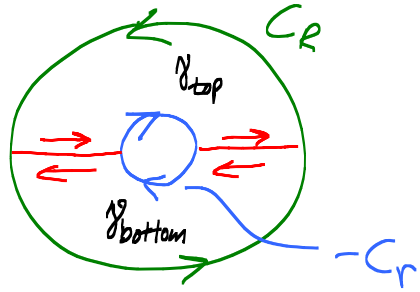
Cauchy Theorem on an annulus



f analytic on a bigger annulus

Then $\left(\int_{C_R} - \int_{C_r}\right) f \, dz = 0$

Why:

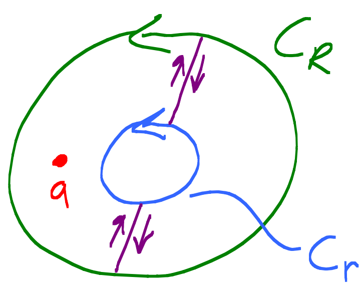


$$\left(\int_{C_R} - \int_{C_r}\right) =$$

$$\left(\int_{C_R} + \int_{-C_r}\right) =$$

$$\underbrace{\int_{\gamma_{top}}}_{0} + \underbrace{\int_{\gamma_{bottom}}}_{0}$$ by Cauchy's Thm

Cauchy Integral Formula for annulus



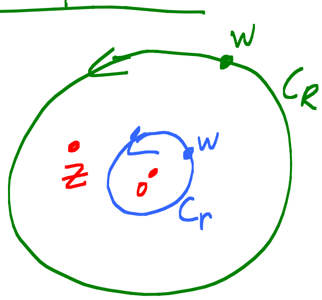
f analytic on bigger annulus.

$$f(a) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{z-a}\, dz$$

$$= \frac{1}{2\pi i}\left(\underbrace{\int_{left}}_{=f(a)} + \underbrace{\int_{right}}_{0}\right) \frac{f(z)}{z-a}\, dz$$

by Cauchy's

Laurent expansions



$$f(z) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(w)}{w-z}\, dw$$

$$= \underbrace{\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\,\frac{1}{1-\frac{z}{w}}\,dw}_{\text{power series!}} - \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{(-z)}\,\frac{1}{1-\frac{w}{z}}\,dw$$

$\left|\frac{w}{z}\right| < 1$ on C_r!

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w^{n+1}}\,dw\right)}_{a_n} z^n \; + \underbrace{\frac{1}{2\pi i\,(z)}\int_{C_r} f(w)\left[1+\left(\frac{w}{z}\right)+\left(\frac{w}{z}\right)^2+\cdots\right]dw}_{\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_r} f(w)\,w^n\,dw\right)\frac{1}{z^{n+1}}}$$

$$\sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i}\int_{C_r} f(w)\,w^n\,dw\right)}_{a_{-(n+1)}} \frac{1}{z^{n+1}}$$

Conclusion:

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\frac{1}{z^n}$$

Laurent expansion

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho} \frac{f(w)}{w^{n+1}}\,dw$ where $r \le \rho \le R$.

<u>Note</u>: $\displaystyle\int_{C_R} = \int_{C_\rho} = \int_{C_r}$

because $\displaystyle\left(\int_{C_R} - \int_{C_\rho}\right) \underbrace{f(w)}_{\text{analytic inside}}\, w^n\,dw = 0$ by Cauchy's

So $\displaystyle\int_{C_R} = \int_{C_\rho}$ ✓

Fact: Laurent series converge nicely:

Both the power series half and $\underbrace{\sum_{1}^{\infty} a_{-n} \frac{1}{z^n}}_{\text{power series in } \frac{1}{z}}$ half

Converge uniformly on $\underbrace{\text{compact sub-annuli}}$.

$$r_1 < \underbrace{r_2 \le |z| \le R_2}_{\text{compact sub-annulus}} < R_1$$

EX: a)
$$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Hmmm. Convergent power series! $R = \infty$. 0 is a removable singularity

b)
$$\frac{\cos z}{z^3} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z^3}$$

$$= \underbrace{\frac{1}{z^3} - \frac{1/2!}{z}}_{\text{only two negative terms } (z^{-n})} + \frac{1}{4!} z + \text{more power series terms}$$

0 is a pole

c)
$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 \leftarrow \text{infinitely many neg. terms.!}$$

0 is an essential singularity.

Big fact: If f is analytic on $D_\varepsilon(z_0) - \{z_0\}$, then f has an "isolated singularity" at z_0.

Good news: Can let C_r shrink down to $r \searrow 0$. So Laurent series is valid on $D_\varepsilon(z_0) - \{z_0\}$.

Huge fact: Only 3 possible behaviors at an isolated singular point:

1) $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$ no neg. terms

Aha! Let $f(0) = a_0$. Convergent power series has a RoC > 0.

z_0 is removable for f.

2) Finitely many $a_{-n} \neq 0$. f has a pole at z_0

Can show that $\lim_{z \to z_0} f(z) = \infty$

3) Infinitely many $a_{-n} \neq 0$. f has an essential singularity at z_0. f shreds the complex plane at z_0! (like $e^{1/z}$ at 0)