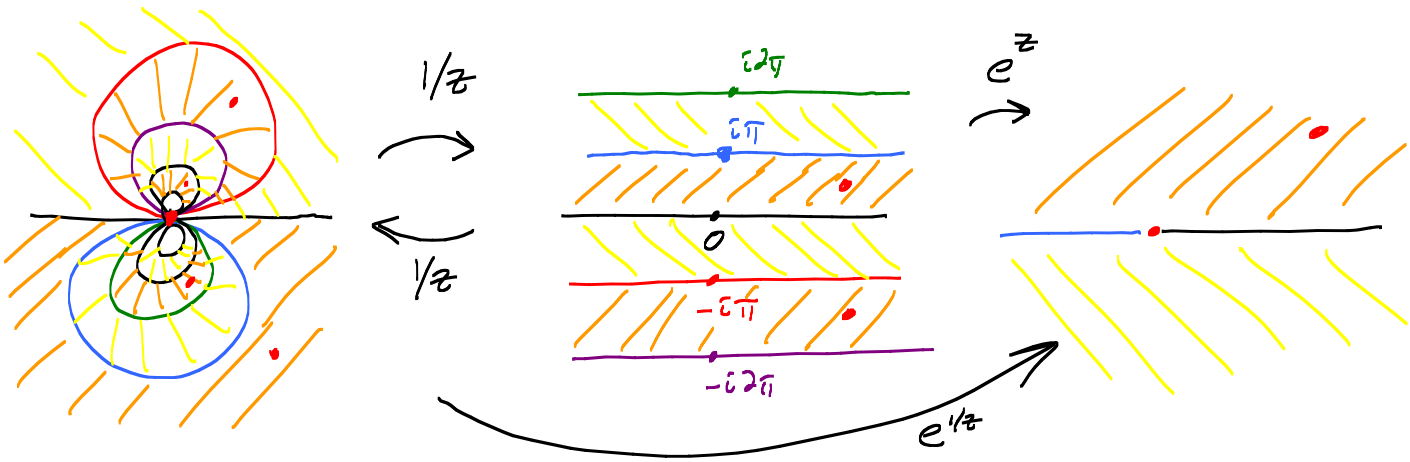
Lesson 25 The Residue Theorem

22, 23, 24 due Wed



Wow! $e^{1/z}$ maps $D_\varepsilon(0)-\{0\}$ onto $\mathbb{C}-\{0\}$ infinite-to-one!

Three kinds of isolated singularities: f analytic on $D_r(z_0)-\{z_0\}$

Removable singularities: 1) No negative terms $a_n \neq 0$ in Laurent expansion.

$\iff$ 2) f can be given a value at z_0 to make it analytic on $D_r(z_0)$.

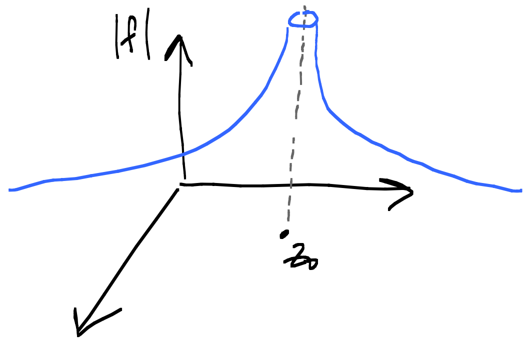
$\iff$ 3) $\lim\limits_{z \to z_0} f(z)$ exists

$\iff$ 4) f is bounded on $D_\rho(z_0)-\{z_0\}$ for $0<\rho<r$.

EX: $\dfrac{\sin z}{z}$ ← make it equal 1 at $z=0$.

Poles: 1) Finitely many negative terms $a_n \neq 0$ in L.E.

$\iff$ 2) $\lim\limits_{z\to z_0} f(z) = \infty$



Residue of f at z_0

$$f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{N-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$= \frac{1}{(z-z_0)^N}\left[\underbrace{a_{-N} + a_{-N+1}(z-z_0) + \cdots + a_0(z-z_0)^N + \cdots}_{\text{regular power series!}}\right]$$

Radius of conv $\geq r$!

Converges to $F(z)$ analytic on $D_r(z_0)$

$$F(z_0) = a_{-N} \neq 0.$$

Wow! Just like zeroes, poles can be "factored out":

$$f(z) = (z-z_0)^{-N} F(z), \quad F(z_0) \neq 0.$$

Note: This shows that $\lim_{z \to z_0} f(z) = \infty$.

EX: $$\frac{\sin z}{z^4} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z^4} = \frac{1}{z^3} - \frac{1/3!}{z} + \frac{1}{5!} z - \cdots$$

Pole of order 3 at $z=0$. $\quad \operatorname{Res}_0 \frac{\sin z}{z^4} = -\frac{1}{3!}$

Essential singularities: 1) Infinitey many neg terms $a_n \neq 0$.

Picards Theorem: Suppose f has an essential sing at z_0. Say it is analytic on $D_r(z_0) - \{z_0\}$.

For any ε with $0<\varepsilon<r$, f maps $D_\varepsilon(z_0)-\{z_0\}$ infinite-to-one onto $\mathbb{C}$, or onto $\mathbb{C}-\{\text{single point}\}$.

EX: $e^{1/z}$ misses the origin.

$$\operatorname{Res}_0 e^{1/z} = 1$$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1/2!}{z^2} + \frac{1/3!}{z^3} + \cdots$$

EX: What kind of singularity does $\cos\left(\frac{1}{z}\right)$ have at $z=0$?

Hmmm. $\cos\left(\frac{1}{t}\right)$ wiggles like crazy as $t \to 0$.

Rules out $z=0$ removable.

Also rules out a pole at $z=0$.

So it must be an essential singularity.

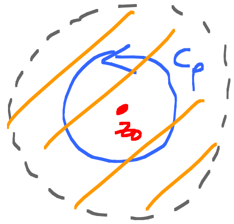
Riemann removable singularity theorem: Suppose f is analytic and bounded on $D_r(z_0)-\{z_0\}$. Then z_0 is a removable singularity.

Why: Can't be a pole. Can't be essential. Must be removable.

Importance of the Residue term:

$$\operatorname{Res}_{z_0} f = a_{-1} = \frac{1}{2\pi i}\int_{C_\rho(z_0)} \frac{f(w)}{w^{-1+1}}\,dw = \frac{1}{2\pi i}\int_{C_\rho} f(w)\,dw$$

Baby residue theorem: $\displaystyle\int_{C_\rho(z_0)} f(w)\,dw = 2\pi i \operatorname{Res}_{z_0} f$



Homework problem:

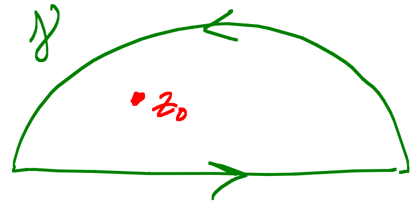
$$h(z) = \frac{A_N}{(z-z_0)^N} + \cdots + \frac{A_2}{(z-z_0)^2} + \frac{A_1}{z-z_0} + g(z)$$

$\mathrm{Res}_{z_0} h$

$\frac{d}{dz} \frac{1}{(-N+1)} (z-z_0)^{-N+1}$

$\frac{d}{dz} \frac{-1}{(z-z_0)}$

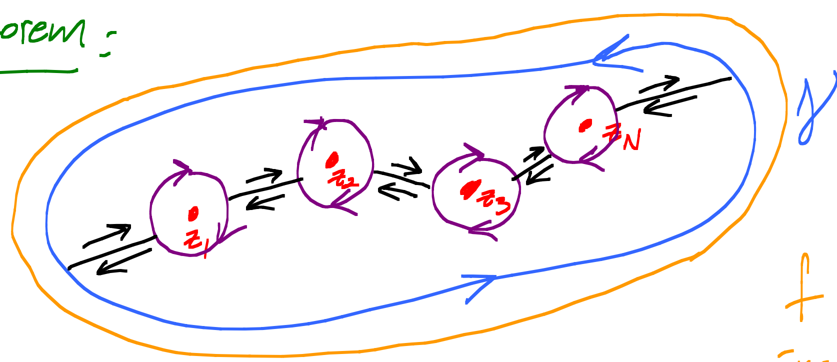
entire



Aha! $\displaystyle \int_\gamma h\, dz = \int_\gamma \frac{A_1}{z-z_0}\, dz = 2\pi i\, A_1$

$$= 2\pi i\, \mathrm{Res}_{z_0} h$$

Residue Theorem:



simple closed curve

f analytic inside and on γ except at finitely many isolated sing. $\{z_j\}_{j=1}^N$

Then $\displaystyle \int_\gamma f\, dz = 2\pi i \sum_{j=1}^N \mathrm{Res}_{z_j} f$

Why: $\displaystyle \left(\int_{\gamma_{top}} + \int_{\gamma_{bot}} \right) f\, dz = \left(\int_\gamma + \sum_{j=1}^N \int_{-C_\varepsilon(z_j)} \right) f\, dz$

So $$\int_\gamma f\,dz = \sum_{j=1}^{N} \underbrace{\int_{C_\varepsilon(z_j)} f\,dz}_{=2\pi i\,\mathrm{Res}_{z_j} f}$$

Important formula: Suppose f and g are analytic on $D_r(z_0)$ and suppose g has a simple zero at z_0. Then $$\mathrm{Res}_{z_0}\frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}.$$

EX: $\dfrac{e^z}{z^2+1}$ at $z=i$.

$f(z) = e^z$

$g(z) = z^2+1$ $\qquad g(i) = 0$

$g'(z) = 2z$ $\qquad g'(i) = 2i \neq 0$

zero of order 1 (simple)

So $$\mathrm{Res}_i \frac{e^z}{z^2+1} = \frac{f(i)}{g'(i)} = \frac{e^i}{2i}$$

Why: $$\frac{e^z}{z^2+1} = \frac{e^z}{(z-i)(z+i)} = \frac{1}{z-i}\Bigg[\underbrace{\frac{e^z}{z+i}}_{\text{analytic near } i}\Bigg]$$

$$= A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \cdots$$

$$= \frac{A_0}{z-i} + A_1 + A_2(z-i) + \cdots$$

$$\mathrm{Res}_i \frac{e^z}{z^2+1} = A_0 = \text{Value of } \frac{e^z}{z+i} \text{ at } i: \frac{e^i}{2i}$$