Lesson 26 Residue integrals

Def^n: Suppose f analytic on $\{z : |z| > R\}$. We think of f as having "an isolated singularity at ∞". The type of the singularity of f at ∞ is the type of the singularity of $f(\frac{1}{z})$ at $z=0$.

EX: 1) $f(z) = z^3$ has a pole of order 3 at ∞.

2) e^z has an ess sing at ∞. $\left[e^{1/z} = 1 + \frac{1}{z} + \frac{1/2!}{z^2} + \cdots \right]$

3) $\frac{z+1}{z-1}$ has a removable sing at ∞.

Fun facts: Suppose f is entire:

∞ is removable $\iff$ $f \equiv$ const.

$\iff$ or f is bounded

∞ is pole $\iff$ f = poly

∞ is essential $\iff$ all other cases.

Picard's little theorem: If f is entire and not a polynomial, then f takes every value in $\mathbb{C}$ infinitely many times with one possible exception. [e^z misses 0.]

Multiplying power series: Just like polynomials!

$$\underbrace{\left(\sum_{n=0}^{\infty} a_n z^n\right)}_{R_a > 0} \underbrace{\left(\sum_{n=0}^{\infty} b_n z^n\right)}_{R_b > 0} = \underbrace{\sum_{n=0}^{\infty} c_n z^n}_{R_c \geq \min(R_a, R_b)}$$

$$C_N = \sum_{k=0}^{N} \underbrace{a_k \, b_{N-k}}_{\text{sum} = N}$$

Why: Leibnitz's formula: $\dfrac{(fg)^{(n)}}{n!} = \displaystyle\sum_{k=0}^{n} \dfrac{\binom{n}{k} f^{(k)} g^{(n-k)}}{n!}$

$$= \sum_{k=0}^{n} \frac{(n!/n!)}{k!\,(n-k)!} f^{(k)} g^{(n-k)}$$

$z=0$: $\quad C_n = \displaystyle\sum_{k=0}^{n} a_k b_{n-k}$ ✓

Applications of residue thm: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$

related to Fourier transform $\displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-isx}\,dx = \hat{f}(s)$

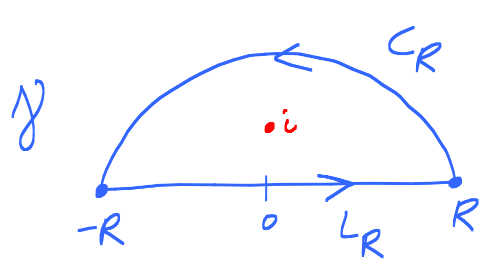
$$\widehat{f'} = is\,\hat{f}$$

Fourier transf: ODEs $\longrightarrow$ algebra

Step 1: $\cos x = \operatorname{Re} e^{ix}$. Let $f(z) = \dfrac{e^{iz}}{z^2+1}$

Notice that $\dfrac{\cos x}{x^2+1} = \operatorname{Re} \dfrac{e^{ix}}{x^2+1}$, $x \in \mathbb{R}$.

Step 2:



$z^2+1=0$ at $z = \pm i$

Res. Thm. $\displaystyle\int_{\gamma} f(z)\,dz = 2\pi i \,\mathrm{Res}_i\, f(z)$

Step 3: Show that $\displaystyle\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

$$\left|e^{i(x+iy)}\right| = \left|e^{ix}e^{-y}\right| = \underbrace{\left|e^{ix}\right|}_{=1}\underbrace{\left|e^{-y}\right|}_{e^{-y}} = e^{-y} \le 1 \text{ when } y \ge 0.$$

So $|e^{iz}| \le 1$ when z is in the Upper Half Plane.

And
$$|z^2+1| = |z^2-(-1)| \underset{\substack{\uparrow\\ \text{denom}\\ \text{estimate}}}{\ge} \Big|\underbrace{|z^2|}_{\substack{R^2\\ \text{on } C_R}} - |-1|\Big| = |R^2-1| \underset{\substack{\uparrow\\ \text{when } R>1}}{=} R^2-1$$

So
$$\left|\int_{C_R} \frac{e^{iz}}{z^2+1}\,dz\right| \le \left(\max_{C_R} \frac{|e^{iz}|}{|z^2+1|}\right) \mathrm{Length}(C_R)$$

$$\le \frac{1}{R^2-1}\cdot \pi R \quad \text{if } R>1.$$

$$\longrightarrow 0 \text{ as } R \to \infty.$$

Step 4: $\displaystyle\int_{L_R} \frac{e^{iz}}{z^2+1}\,dz = \int_{-R}^{R} \frac{e^{it}}{t^2+1}\cdot 1\cdot dt$ $\qquad L_R: z(t) = t$, $-R \le t \le R$

$$= \int_{-R}^{R} \frac{\cos t}{t^2+1}\,dt + i\int_{-R}^{R} \underbrace{\frac{\sin t}{t^2+1}}_{\text{odd!}}\,dt$$

zero!

Step 5: $\displaystyle \operatorname{Res}_i \frac{e^{iz}}{z^2+1}$ $\begin{array}{l} \leftarrow f(z) \\ \leftarrow g(z) \end{array}$

$g(i) = 0$ ✓

$g'(i) = 2i \neq 0$

g has a simple zero at i

$$= \frac{f(i)}{g'(i)} = \frac{e^{i \cdot i}}{2i} = \frac{e^{-1}}{2i}$$

Step 6: $\displaystyle \int_\gamma f\,dz = 2\pi i \operatorname{Res}_i f$ $\leftarrow R > 1$

$$\underbrace{\int_{L_R} f\,dz}_{} + \underbrace{\int_{C_R} f\,dz}_{} = 2\pi i \left(\frac{e^{-1}}{2i} \right)$$

Let $R \to \infty$.

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\,dt + 0 = \frac{\pi}{e}$$

Fact: $\displaystyle \operatorname{Res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$ when g has simple zero at z_0.

why: $g(z) = \underbrace{g(z_0)}_{\substack{= \\ 0}} + \underbrace{g'(z_0)}_{\neq 0}(z-z_0) + a_2(z-z_0)^2 + \cdots$

$a_0 = 0$ $\qquad a_1 = g'(z_0)$

$$= (z-z_0)\underbrace{\left[a_1 + a_2(z-z_0) + \cdots \right]}_{G(z) \text{ analytic. } G(z_0) = a_1}$$

$\boxed{G(z_0) = g'(z_0)}$

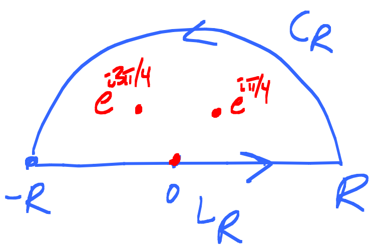
$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-z_0)G(z)} = \frac{1}{z-z_0}\left[\frac{f(z)}{\underbrace{G(z)}_{A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \cdots}} \right]$$

$$= \frac{A_0}{z - z_0} + A_1 + A_2(z - z_0) + \cdots$$

$A_0 \leftarrow Res_{z_0}$

$$\text{So } \quad Res_{z_0} \frac{f}{g} = A_0 = \frac{f(z_0)}{G(z_0)} = \frac{f(z_0)}{g'(z_0)} \quad \checkmark$$

EX: Find $\displaystyle I = \int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$



$$f(z) = \frac{1}{z^4+1}$$

$g(z) = z^4 + 1$

$g(e^{i\pi/4}) = 0 \checkmark$

$g'(z) = 4z^3$

$g'(e^{i\pi/4}) \neq 0.$

$$Res_{e^{i\pi/4}} \frac{1}{z^4+1} = \frac{1}{4(e^{i\pi/4})^3} = \frac{1}{4} e^{-i3\pi/4}$$

$$\left| \int_{C_R} \frac{1}{z^4+1}\, dz \right| \leq \frac{1}{R^4-1} \cdot \pi R \quad \text{if } R > 1.$$

$$\longrightarrow 0 \quad \text{as } R \to \infty.$$

$$\int_{L_R} + \int_{C_R} = 2\pi i \left(Res_{e^{i\pi/4}} f + Res_{e^{i3\pi/4}} f \right)$$

$R \to \infty$:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx + 0 = 2\pi i \left(\frac{1}{4(e^{i\pi/4})^3} + \frac{1}{4(e^{i3\pi/4})^3} \right)$$

$$= \frac{\pi}{2} i \left(\underbrace{e^{-i3\pi/4}}_{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i} + \underbrace{e^{-i9\pi/4}}_{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i} \right) = \frac{\pi}{\sqrt{2}}$$

ans!

Book:
$$\frac{1}{z^4+1} = \frac{1}{(z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$

$$= \frac{1}{z-e^{i\pi/4}}\left[\frac{1}{\underbrace{(\quad)(\quad)(\quad)}_{H(z)}}\right]$$

$$\mathrm{Res}_{e^{i\pi/4}} = H(e^{i\pi/4}) \qquad \text{Ugh!}$$