Lesson 27 More residue integrals

25, 26, 27 due Wed, Exam 2 in class Wed, Nov 13 covering Lessons 16–32

Fact: f, g analytic near z_0. g has a simple zero at z_0: $g(z_0)=0$, $g'(z_0)\neq 0$.

Then $$\operatorname{Res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$$

Easy case: $g(z) = z - z_0$. Then $\operatorname{Res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{1} = f(z_0)$

Fun thing: $$\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i \operatorname{Res}_{z_0} \frac{f}{z-z_0} = 2\pi i f(z_0). \quad \text{(Cauchy!)}$$

Parallel facts about zeroes and poles

Zeroes

f has a zero of order N at z_0

$f(z_0)=0, f'(z_0)=0, \dots, f^{(N-1)}(z_0)=0$

but $f^{(N)}(z_0)\neq 0$.

$$f(z) = a_N (z-z_0)^N + \cdots$$
$$= (z-z_0)^N \underbrace{\left[a_N + a_{N+1}(z-z_0) + \cdots\right]}_{F(z)}$$

$\boxed{f(z) = (z-z_0)^N F(z)}$ where F is analytic near z_0 and $F(z_0)\neq 0$

Poles

f has a pole of order N at z_0

$$f(z) = \frac{A_N}{(z-z_0)^N} + \cdots + \frac{A_1}{z-z_0} + \text{power series}$$
$$= \frac{1}{(z-z_0)^N} \underbrace{\left[A_N + A_{N-1}(z-z_0) + \cdots\right]}_{F(z)\text{ analytic near } z_0}$$

$F(z_0) = A_N \neq 0$

$\boxed{f(z) = (z-z_0)^{-N} F(z)}$

Today's big fact: Suppose $P(z)$ and $Q(z)$ are polynomials and

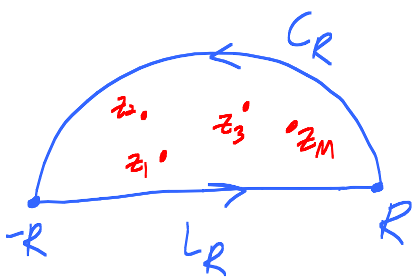
1) Q has no zeroes on $\mathbb{R}$, and

2) $\deg Q \geq \deg P + 2$

Then $$\int_{-\infty}^{\infty} \frac{P(t)}{Q(t)}\,dt = 2\pi i \sum_{\substack{z_k \text{ zeroes} \\ \text{of } Q \\ \text{in UHP}}} \operatorname{Res}_{z_k} \frac{P}{Q}$$

Also $$\int_{-\infty}^{\infty} \frac{P(t)}{Q(t)}\,e^{it}\,dt = 2\pi i \sum_{\substack{\text{zeroes} \\ \text{of } Q \\ \text{in UHP}}} \operatorname{Res}_{z_k} \frac{P}{Q}e^{iz}$$

Why:



Key: Basic polynomial estimate

$$P(z) = a_N z^N + \cdots + a_0$$

$$\boxed{A|z|^N \le |P(z)| \le B|z|^N, \quad |z| > R.}$$

denom est for Q numerator est for P

$$\left| \int_{C_R} \frac{P}{Q}\,dz \right| \le \left(\max_{z \in C_R} \left| \frac{P(z)}{Q(z)} \right| \right) \text{Length}(C_R)$$

$$\le \left(\frac{B_P R^{\deg P}}{A_Q R^{\deg Q}} \right) \pi R \qquad \text{when } R > \max(R_P, R_Q)$$

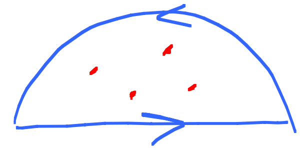
$$\le (\text{const})\, R^{\deg P + 1 - \deg Q} \leftarrow \text{a negative number!}$$

$$\longrightarrow 0 \quad \text{as } R \to \infty.$$

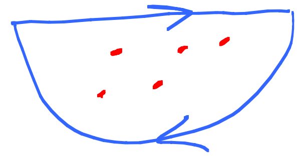
$$\underbrace{\int_{L_R}}_{\downarrow \text{ want}} + \underbrace{\int_{C_R}}_{\downarrow\ 0} = 2\pi i \sum_{\text{UHP}} \operatorname{Res} \frac{P}{Q}$$

$$= \text{algebra!}$$

Remark: $|e^{iz}|$ is bounded in UHP.



$|e^{-iz}|$ is bounded in LHP.



clockwise!

$$\int_{-\infty}^{\infty} \frac{P}{Q} e^{-iz}\, dz = -2\pi i \sum_{\substack{\text{zeroes } Q \\ \text{in LHP}}} \operatorname{Res} \frac{P}{Q} e^{-iz}$$

Trig integrals: $$I = \int_0^{2\pi} \cos^6 t \, dt$$

Hmmm: $C: \ z(t) = e^{it}, \quad 0 \le t \le 2\pi$

$dz = z'(t)\, dt = ie^{it}\, dt$

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{1}{2}\left[e^{it} + \frac{1}{e^{it}}\right] = \frac{1}{2}\left[z(t) + \frac{1}{z(t)}\right]$$

$$I = \int_0^{2\pi} \left(\frac{1}{2}\left[z(t) + \frac{1}{z(t)}\right]^6\right) \frac{\overbrace{ie^{it}\, dt}^{dz}}{\underbrace{ie^{it}}_{=iz(t)}}$$

Aha! $$I = \int_C \underbrace{\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^6 \frac{1}{iz}}_{f(z)} dz$$

$$= 2\pi i \operatorname{Res}_0 f(z)$$

Residue term!

where $$f(z) = \frac{1}{2^6} \cdot \frac{1}{iz}\left(z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^6} + \frac{1}{z^6}\right)$$

$$I = 2\pi i \, \frac{20}{2^6 i} = \frac{5\pi}{8}$$

General method: $\displaystyle\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

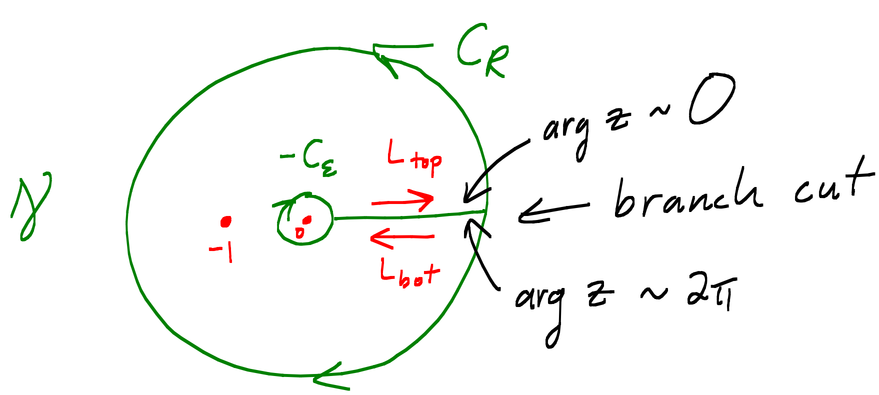
R: rational, does not blow up on C

$$= \int_C R\left(\frac{z(t) + \frac{1}{z(t)}}{2}, \frac{z(t) - \frac{1}{z(t)}}{2i} \right) \underbrace{\frac{1}{iz}\, dz}_{= dt}$$

now use the Residue thm.

Prob: Compute $\displaystyle I = \int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\, dx$

$$f(z) = \frac{1}{z^{1/2}(z+1)}$$



$$z^{1/2} = e^{\frac{1}{2}\log_0 z}$$

where $\log_0 z = \mathrm{Ln}|z| + i\theta$

where $\theta \in \arg z$, $0 < \theta < 2\pi$

$$\int_\gamma f\, dz = 2\pi i \, \mathrm{Res}_{-1} f = 2\pi i \cdot \frac{1}{(-1)^{1/2}}$$

$$= 2\pi i \, \frac{1}{e^{\frac{1}{2}(0 + i\pi)}} = \frac{2\pi i}{\underbrace{e^{i\pi/2}}_{= i}} = 2\pi$$

$$\int_{L_{top}} f\,dz = \int_{\varepsilon}^{R} \frac{1}{\sqrt{t}\,(t+1)}\,dt \leftarrow \text{want!}$$

$$\int_{L_{bot}} f\,dz = -\int_{\varepsilon}^{R} \frac{1}{\underbrace{e^{\frac{1}{2}(\operatorname{Ln} t + i2\pi)}}_{\sqrt{t}\,\underbrace{e^{i\pi}}_{-1}}\,(t+1)}\,dt = \int_{\varepsilon}^{R} \frac{1}{\sqrt{t}\,(t+1)}\,dt$$

$\uparrow$ want!

Last steps: $\left|\int_{C_R} f\,dz\right|, \left|\int_{-C_\varepsilon} f\,dz\right|$ both go to 0

(as $R \to \infty$) (as $\varepsilon \to 0$)

why: $\left|z^{1/2}\right| = \left|e^{\frac{1}{2}(\operatorname{Ln}|z| + i\theta)}\right| = \left|e^{\frac{1}{2}\operatorname{Ln}|z|}\, e^{i\frac{\theta}{2}}\right| = |z|^{1/2}$

$$\left|\int_{C_R} f\,dz\right| \le \frac{1}{R^{1/2}(R-1)} \cdot 2\pi R \quad \text{when } R > 1$$

$$\to 0 \text{ as } R \to \infty.$$

on $-C_\varepsilon$ $\quad |z+1| \ge \big|\,|z| - |-1|\,\big| = |\varepsilon - 1| = 1 - \varepsilon$ if $\varepsilon < 1$.

$$\left|\int_{-C_\varepsilon} f\,dz\right| \le \frac{1}{\varepsilon^{1/2}(1-\varepsilon)} \cdot 2\pi\varepsilon \quad \text{if } 0 < \varepsilon < 1$$

$$\to 0 \text{ as } \varepsilon \to 0.$$

$$\left(\int_{L_{top}} + \int_{C_R} + \int_{L_{bot}} + \int_{-C_\varepsilon}\right) f\,dz = 2\pi$$

Let $R \to \infty$.
Then let $\varepsilon \to 0$.
Get $2I = 2\pi$.
$I = \pi$