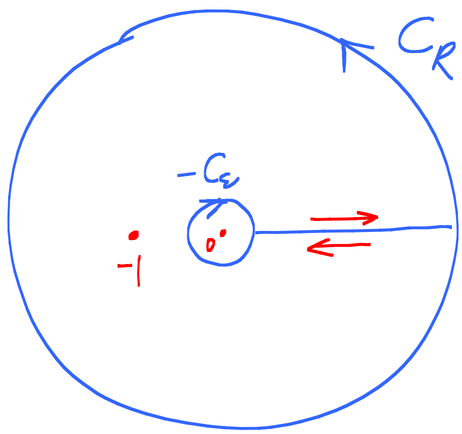
Lesson 28

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

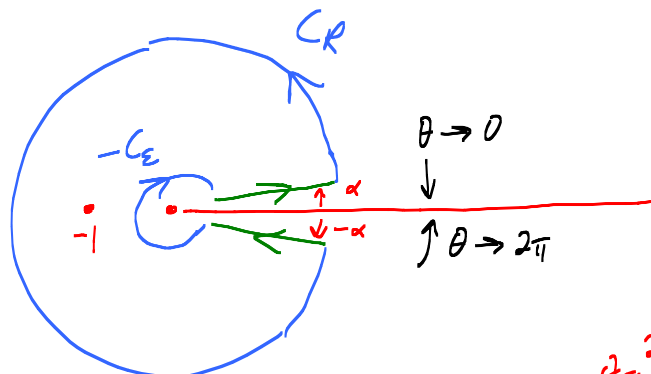


On C_R: $|z+1| \geq \big|\,|z| - |-1|\,\big| = R-1$ if $R>1$

C_ε: $|z+1| \geq |\varepsilon - 1| = 1-\varepsilon$ if $\varepsilon < 1$

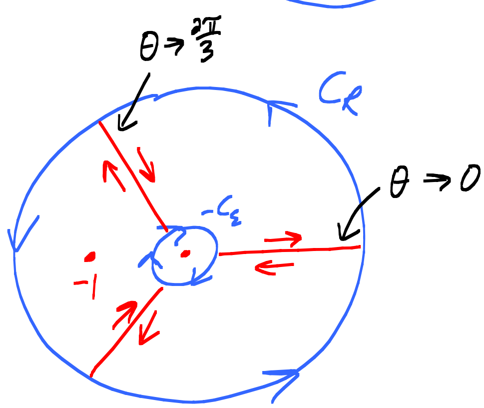
$$\int_0^\infty \frac{1}{\sqrt{t}\,(t+1)}\, dt = \pi$$

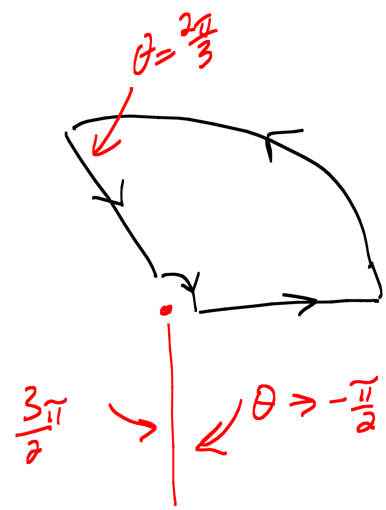
More carefully:



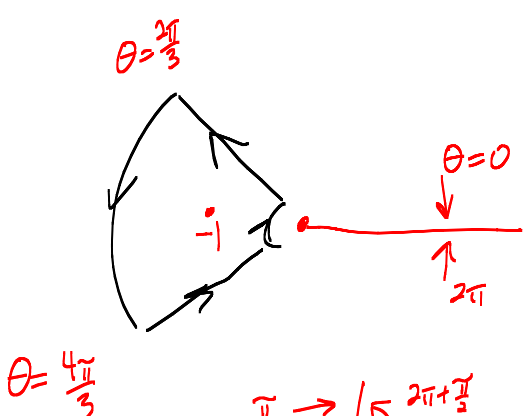
Let $\alpha \to 0$.
$R \to \infty$
$\varepsilon \to 0$

Or:

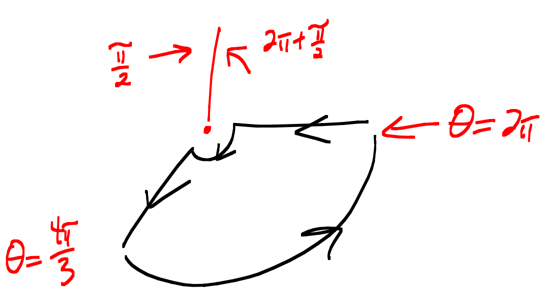




$\int_\gamma f\, dz = 0$ by Cauchy



$\int_\gamma f\, dz = 2\pi i \operatorname{Res}_{-1} f$



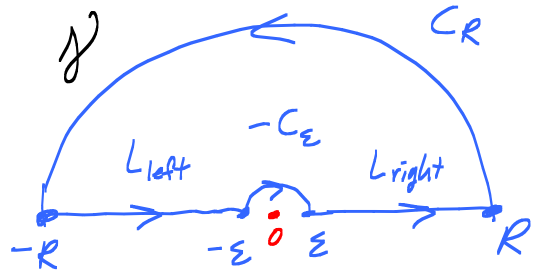
$\int_\gamma f\, dz = 0$ by Cauchy's

$$z^{1/2} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}(\ln t + i\theta)} = \begin{cases} \sqrt{t} & \theta = 0 \\ e^{i\pi}\sqrt{t} = -\sqrt{t} & \theta = 2\pi \end{cases}$$

EX: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$

Need to use $\sin t = \operatorname{Im} e^{it}$

$$f(z) = \frac{e^{iz}}{z}$$



Cauchy's Thm: $\displaystyle\int_\gamma f\, dz = 0$

Step 1: $$\left(\int_{L_{left}} + \int_{L_{right}}\right) f(z)\,dz = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \frac{e^{it}}{t}\, dt$$

$$\frac{e^{it}}{t} = \underbrace{\frac{\cos t}{t}}_{\text{odd}} + i \underbrace{\frac{\sin t}{t}}_{\text{even}}$$

$$= i\,2\int_{\varepsilon}^{R} \frac{\sin t}{t}\, dt$$

Step 2: $\displaystyle\int_{C_R} f\, dz \to 0$ as $R \to \infty$.

Basic estimate fails us:

$|e^{iz}| \le 1$ when $\operatorname{Im} z \ge 0$

$$\left|\int_{C_R} \frac{e^{iz}}{z}\, dz\right| \le \left(\max_{C_R} \left|\frac{e^{iz}}{z}\right|\right) \operatorname{Length}(C_R)$$

$$\leq \frac{1}{R} \cdot \pi R = \pi \quad \text{Ouch!}$$

Need Jordan's lemma: $C_R : z(t) = Re^{it}, \; 0 \leq t \leq \pi$

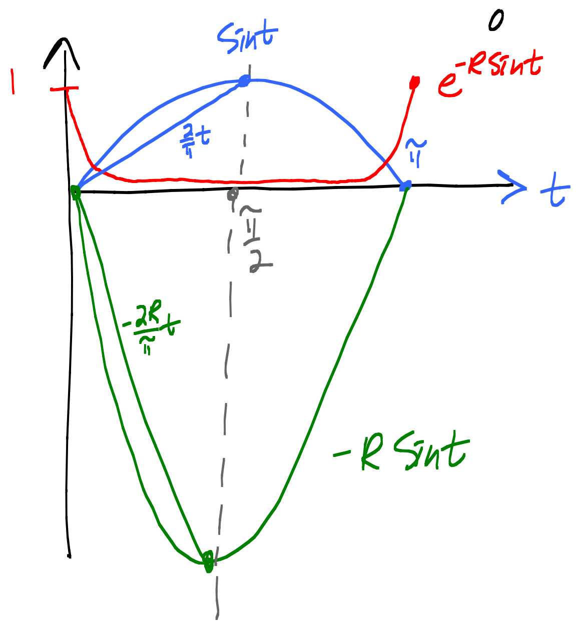
$z'(t) = iRe^{it}$

$$\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| = \left| \int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}} \underbrace{iRe^{it} dt}_{dz} \right|$$

$$= \left| i \int_0^{\pi} e^{i(R\cos t + iR\sin t)} \right| dt$$

$$\leq \int_0^{\pi} \underbrace{\left| e^{iR\cos t} \right|}_{=1} \left| e^{-R\sin t} \right| dt$$

$$\leq \int_0^{\pi} e^{-R\sin t}\, dt \leftarrow \text{need} \to 0 \text{ as } R \to \infty.$$



(Can see the area under $e^{-R\sin t} \to 0$ as $R \to \infty$ using very elementary arguments.)

$0 \leq \frac{2}{\pi} t \leq \sin t$ on $0 \leq t \leq \frac{\pi}{2}$

$$-R\sin t \leq -\frac{2R}{\pi} t \leq 0$$

$$\boxed{e^{-R\sin t} \leq e^{-\frac{2R}{\pi} t}} \leq 1$$

Now $\displaystyle\int_0^{\pi} e^{-R\sin t}\,dt = 2\int_0^{\pi/2} e^{-R\sin t}\,dt$

$$\leq 2\int_0^{\pi/2} e^{-\frac{2R}{\pi}t}\,dt$$

$$= 2\left[-\frac{\pi}{2R}e^{-\frac{2R}{\pi}t}\right]_0^{\pi/2}$$

$$= \frac{\pi}{R}\left(1-e^{-R}\right) < \frac{\pi}{R} \to 0$$

as $R \to \infty$

Last step: $\displaystyle\int_{-C_\varepsilon} f\,dz \to ?$

$$\frac{e^{iz}}{z} = \frac{\left(1+\frac{iz}{1!}+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\cdots\right)}{z}$$

$$= \frac{1}{z} + \underbrace{i - \frac{1}{2!}z + \text{power series}}_{g(z)\ \text{entire!}}$$

Bounded by M on $\overline{D_1(0)}$.

Now $\displaystyle\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz = \int_{-C_\varepsilon} \frac{1}{z}\,dz + \int_{-C_\varepsilon} g(z)\,dz$

$$\downarrow \qquad\qquad \downarrow$$

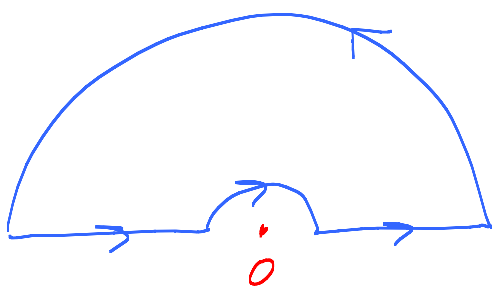
$$-i\pi \quad + \quad 0$$

$C_\varepsilon : z(t) = \varepsilon e^{it}, \; 0 \le t \le \pi$

$$\int_{-C_\varepsilon} \frac{1}{z}\, dz = -\int_{C_\varepsilon} \frac{1}{z}\, dz = -\int_0^{\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt = \underline{\underline{-i\pi}}$$

and $$\left| \int_{-C_\varepsilon} g\, dz \right| \le \underbrace{\left(\underset{C_\varepsilon}{\text{Max}} |g| \right)}_{\le M} \underbrace{\text{Length}(C_\varepsilon)}_{\pi\varepsilon} \quad \text{if } 0 < \varepsilon < 1$$

$\longrightarrow 0$ as $\varepsilon \to 0$.



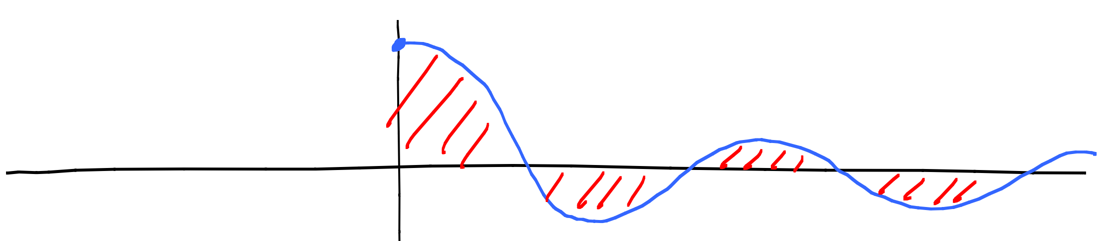
$$\int_{C_R} + \underbrace{\left(\int_{\text{left}} + \int_{\text{right}} \right)}_{} + \int_{-C_\varepsilon} = 0$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$0 + i2\int_0^\infty \frac{\sin t}{t}\, dt + (-i\pi) = 0$$

Get $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$ and $\displaystyle\int_{-\infty}^\infty \frac{\sin t}{t}\, dt = \pi$

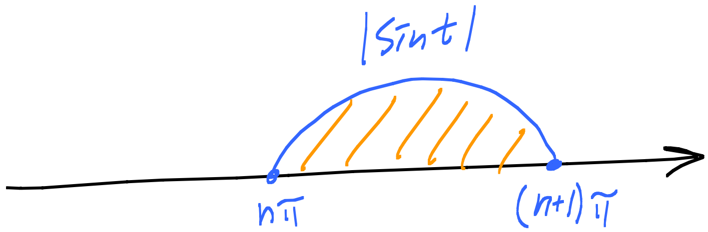
<u>Famous</u> $\int$: Conditionally convergent $\int$



Areas of humps $\to 0$
$\pm$

Alternating series test shows $\lim_{R\to\infty}\int_0^R \frac{\sin t}{t}\,dt$ exists.

but $\int$ is not absolutely integrable.



area = 2

$$\int \frac{|\sin t|}{(n+1)\pi} \le \int \frac{|\sin t|}{t} \le \int \frac{|\sin t|}{n\pi}$$

$$\frac{2}{(n+1)\pi} \le \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{t}\,dt \le \frac{2}{n\pi}$$

$\sum$ diverges! $\qquad \sum_{1}^{\infty} \frac{1}{n} = \infty$