# Lesson 30 The Argument Principle

28, 29, 30 due Wed
Exam 2 in class, Wed, Nov. 13

Great problem:

$f(z)=e^{-z^2}$ Know $\int_0^\infty e^{-x^2}dx$

$\int_T f(z)\,dz$ gives Fresnel integrals $\int_0^\infty \sin x^2\,dx$ and $\cos x^2$.

**Fact:**

$$\text{Res}_{z_0}\frac{f'}{f} = \begin{cases} 0 & \text{if } f \text{ analytic},\ f(z_0)\neq 0 \\ N=\text{order of } \underline{\text{zero}} \text{ if } f(z_0)=0 \\ -N, & N \text{ order of pole at } z_0 \end{cases}$$

Why:

$$f(z)=a_N(z-z_0)^N+\cdots = (z-z_0)^N\Big[\underbrace{a_N + a_{N+1}(z-z_0)+\cdots}_{F(z),\ F(z_0)=a_N\neq 0}\Big]$$

$$= (z-z_0)^N F(z)$$

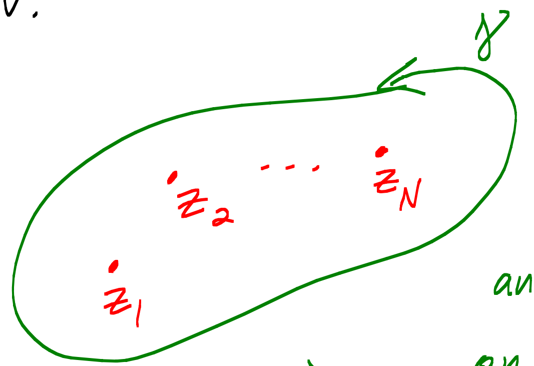
So

$$\frac{f'}{f} = \frac{N(z-z_0)^{N-1}F(z)}{(z-z_0)^N F(z)} + \frac{(z-z_0)^N F'(z)}{(z-z_0)^N F(z)}$$

$$\frac{f'(z)}{f(z)} = \frac{\overset{\text{Res}}{\boxed{N}}}{z-z_0} + \boxed{\frac{F'(z)}{F(z)}} \leftarrow \text{analytic near } z_0$$

Note: Exactly same calculation shows $\text{Res}_{z_0}\frac{f'}{f}=-N$ at a pole of order $N$.

**Argument Principle:**



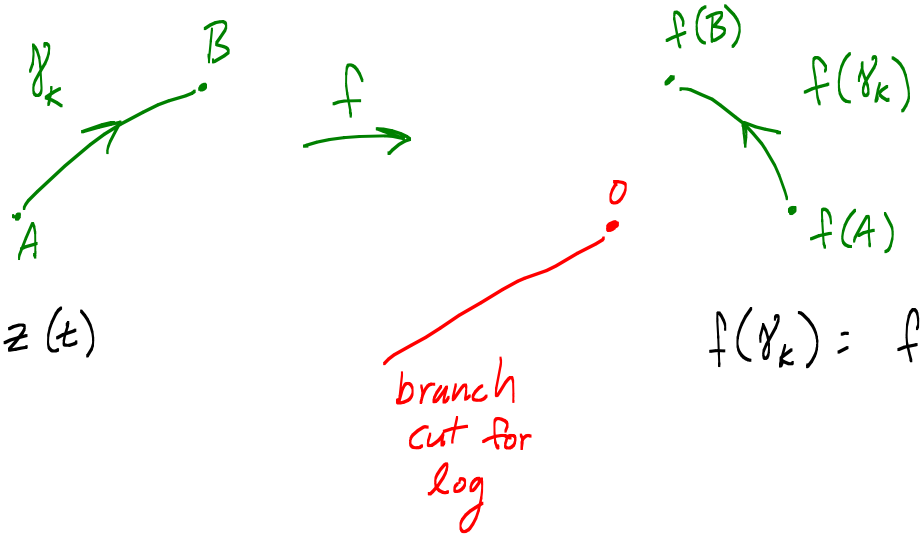
$f$ analytic inside and on $\gamma$ and non-vanishing on $\gamma$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \left(\begin{array}{c}\text{Total \# of zeroes}\\ \text{of } f \text{ inside } \gamma\\ \text{counted with mult.}\end{array}\right)$$

Pf: Residue thm

Remark: Allow poles: $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{matrix}\#\text{ zeroes}\\ f \text{ in } \gamma\\ \text{with mult}\end{matrix}\right) - \left(\begin{matrix}\#\text{ poles of}\\ \text{in } \gamma\\ \text{with order}\end{matrix}\right)$

Why "Argument" principle? $\frac{f'(z)}{f(z)} = \frac{d}{dz}\left[\log f(z)\right]$



$\gamma_k : z(t)$

$f(\gamma_k) : f(z(t))$

Now $\int_{\gamma_k} \frac{f'}{f}\,dz = \int_{\gamma_k} \frac{d}{dz}(\log f)\,dz = \log f(B) - \log f(A)$

$$= \left[\operatorname{Ln}|f(B)| + i\arg f(B)\right] - \left[\operatorname{Ln}|f(A)| + i\arg f(A)\right]$$

$$= \underbrace{\left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)|\right)} + i\,\Delta_{\gamma_k}\arg f$$

← does not depend on branch!

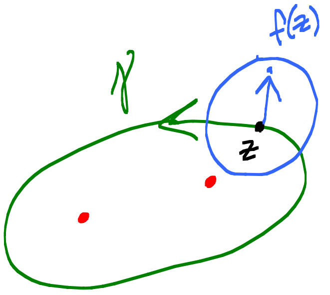
Aha! Add up $\int_{\gamma_k}$'s around $\gamma$. Get pairwise cancelation, first term cancels last term!

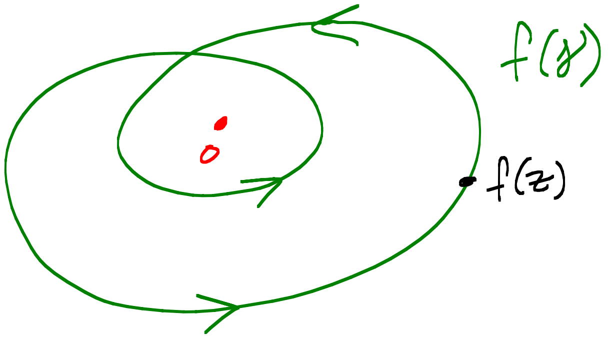
Improved version:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \begin{pmatrix}\text{Total \#}\\ \text{zeroes}\\ \text{of in}\\ \gamma\end{pmatrix} = \frac{1}{2\pi i}\, i\, \Delta_\gamma \arg f$$

$$= \begin{pmatrix}\text{\# times } f(z) \text{ spins around}\\ \text{the origin as } z \text{ goes around } \gamma\end{pmatrix}$$

EX:





f(z) spins around 0 twice as z goes around γ

Conclude: f has two zeroes inside γ.

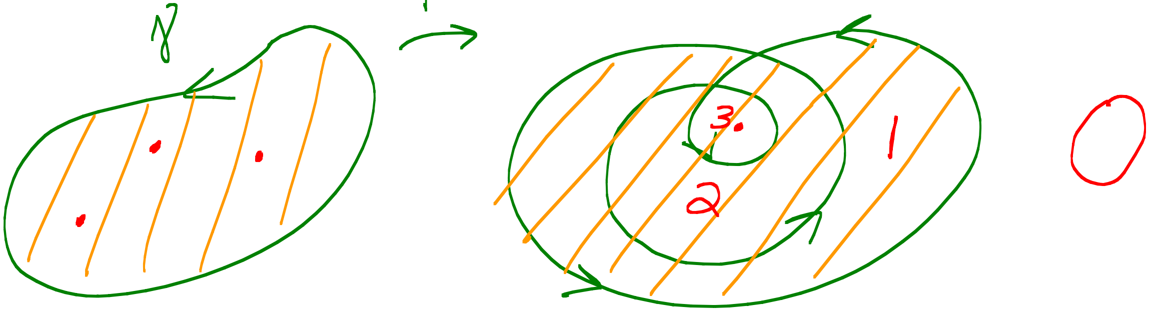
Two possibilities: 2 simple zeroes, or
1 double zero.

Great fact: Replace $\dfrac{f'(z)}{f(z)}$ by $\dfrac{f'(z)}{\underbrace{f(z)-w_0}_{F(z)}} \longleftarrow F'(z)$

Def$^n$: Say that $f(z)=w_0$ with multiplicity $m$ at $z=z_0$ exactly when $f(z)-w_0$ has a zero of mult $m$ at $z_0$.

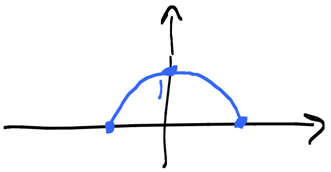
EX: $f(z) = 2+(z-1)^3$. $f(1)=2$ with mult 3

Picture:



**Open mapping theorem:** A non-constant analytic fcn maps open sets to open sets.

Way false $\mathbb{R} \to \mathbb{R}$: $h(t) = 1 - t^2$



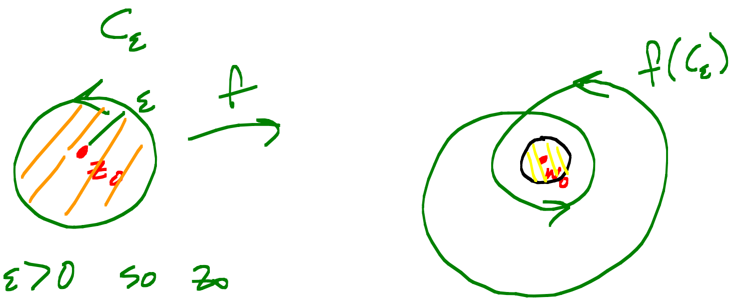
$h : (-1, 1) \to (0, 1]$

open　　not open!

**Why Open mapping thm is true:** $f(z_0) = w_0$. $f$ not constant.

Then $f(z) - w_0 = (z - z_0)^m F(z)$

↑ isolated zero. $F(z_0) \neq 0$



Shrink $\varepsilon > 0$ so $z_0$ is only zero of $f(z) - w_0$ inside $C_\varepsilon$.

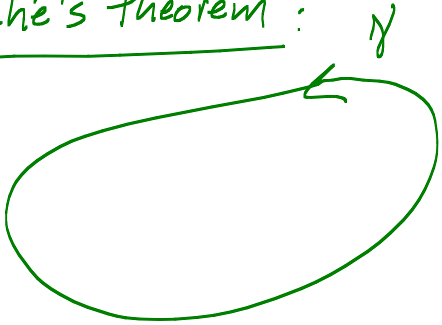
Yellow disc all gets hit by $f$ at least once.

Conclude $f(\text{Disc about } z_0)$ contains a disc about $w_0$:

So $f : (\text{open sets}) \longrightarrow (\text{open sets})$.

[ $f$ continuous : $f^{-1}(\text{open set}) = \text{open set}$ ]

**Rouché's theorem:**

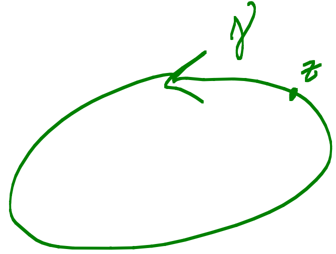


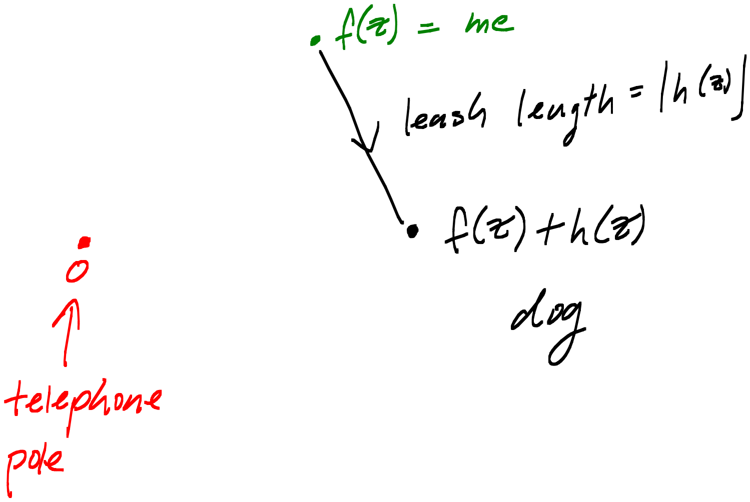
$f$ and $h$ analytic inside and on $\gamma$ (counterclockwise, simple).

If $|h(z)| < |f(z)|$ along $\gamma$,

then $f$ and $f + h$ have same # zeroes (counted with mult) inside $\gamma$.

# Feeling





$$\underbrace{|h(z)|}_{\text{leash length}} < \underbrace{|f(z)|}_{\text{dist. me to pole}}$$

Dog goes around pole same # times as I do.