

Lesson 31 Rouché's Theorem

Rouché's Theorem: Suppose f and h are analytic inside and on a simple closed curve γ and $|h(z)| < |f(z)|$ on γ . Then f and $f+h$ have same number of zeroes (counted with mult) inside γ .

EX: $\underbrace{z^3 + 9z}_{h(z)} + \underbrace{27}_{f(z) \leftarrow \text{the biggest}}$ on $D_2(0)$. How many zeroes

Note: $|h(z)| \leq |z^3| + |9z| = \underbrace{2^3 + 9 \cdot 2}_{26 < 27 = |f(z)|}$ on $C_2(0)$.
on $C_2(0)$.

So $|h(z)| < |f(z)|$ on $C_2(0)$.

Rouché's $\Rightarrow f$ and $f+h$ have same # zeroes inside, f has none. So $z^3 + 9z + 27$ also has none.

EX: Show that all the zeroes of $z^6 - 5z^2 + 10$ fall in the annulus $\{z: 1 < |z| < 2\}$.

On $C_2(0)$: $z^6 - 5z^2 + 10$

$$\uparrow \\ |z^6| = 2^6 = 64$$

on $C_2(0)$
Big one

$$f(z) = z^6$$

$$h(z) = -5z^2 + 10$$

$$\text{Then } |h(z)| \leq |5z^2| + 10$$

$$= 5 \cdot 2^2 + 10 = 30$$

$$< 64 = |f(z)| \text{ on } C_2$$

f has 6 zeroes inside $C_2(0)$. Hence, so does $f+h = z^6 - 5z^2 + 10$.

[f only has 6 zeroes on $\mathbb{C}$ by Fund. Thm. Alg.]

On $C_1(0)$: $z^6 - 5z^2 + \underbrace{10}_{=f(z)} \leftarrow \text{the big one}$

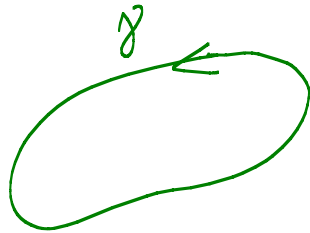
$$\begin{aligned} \text{Then } h(z) = z^6 - 5z^2 \text{ and } |h(z)| &\leq |z^6| + |5z^2| \\ &= 1^6 + 5 \cdot 1^2 \\ &= 6 < 10 = f(z) \text{ on } C_1 \end{aligned}$$

So $|h| < |f|$ on $C_1(0)$. f has no zeroes. So

$z^6 - 5z^2 + 10 = f+h$ also has none.

So all 6 zeroes of poly fall inside C_2 and outside C_1 . ✓

Why Rouché's works:

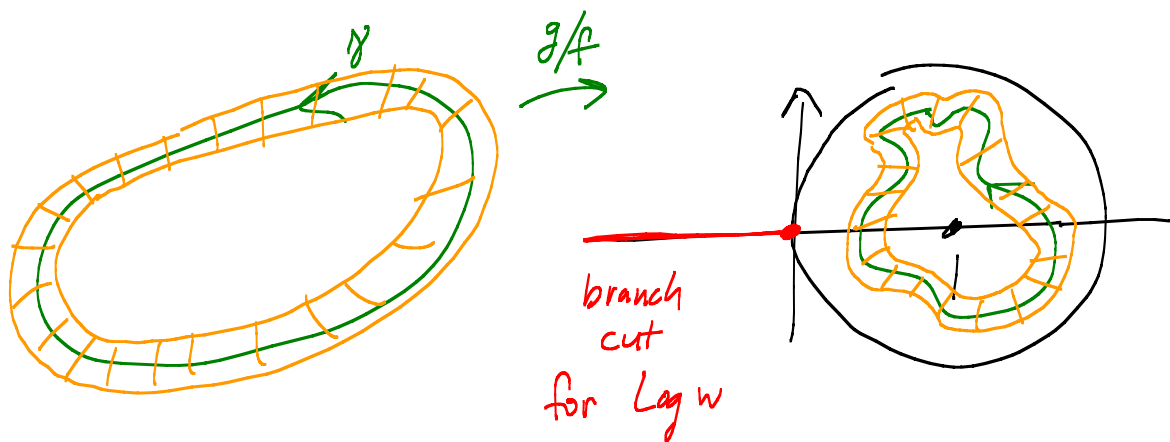


(*) $|h| < |f|$ on γ .

Let $\underbrace{g=f+h}$ $\leftarrow$ want to show g & f have same # zeroes.
Divide by f

$$\frac{g}{f} = 1 + \frac{h}{f}$$

$$\left| 1 - \frac{g}{f} \right| = \left| \frac{h}{f} \right| < 1 \text{ on } \gamma \text{ by } (*)$$



Cool thing: $F = \frac{g}{f}$. Then $\frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f}$

Aha! $\frac{1}{2\pi i} \int_{\gamma} \frac{F'}{F} dz = \underbrace{\frac{1}{2\pi i} \int_{\gamma} \frac{g'}{g} dz}_{\substack{\# \text{ zeroes} \\ g}} - \underbrace{\frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} dz}_{\substack{\# \text{ zeroes} \\ f}}$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{d}{dz} (\text{Log } F) dz$$

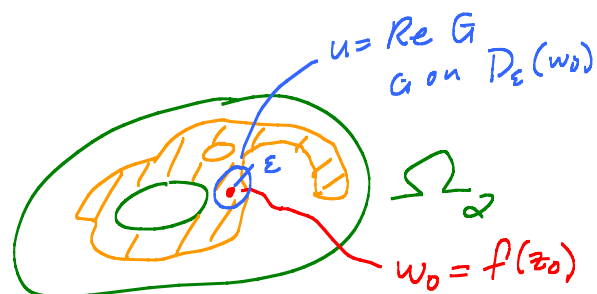
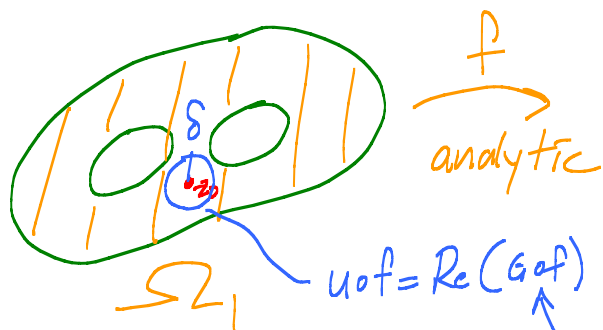
$$= 0$$

Start reading Chap 7.

Fact: u is harmonic on a domain Ω if and only if it is locally the real part of an analytic fcn, meaning that, for each $z_0 \in \Omega$, there is a disc $D_{\varepsilon}(z_0) \subset \Omega$ and an analytic fcn f on $D_{\varepsilon}(z_0)$ such that $u = \text{Re } f$ on $D_{\varepsilon}(z_0)$.

Note: If Ω is simply connected, then we can get an analytic f on all of Ω such that $u = \text{Re } f$ on Ω .

Thm: (7.1)



u harmonic on Ω_2

δ, ε : Continuity of analytic f at z_0

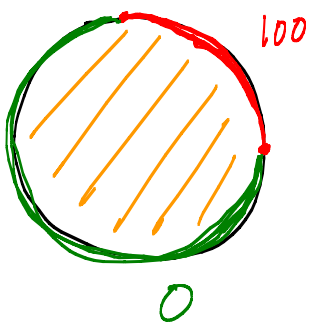
Then $u \circ f$ is harmonic on Ω_1 .

"Harmonic composed with analytic are harmonic."

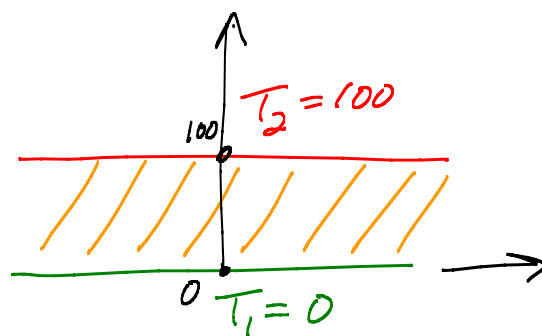
The hard way: $\Delta(u \circ f) = \dots$ chain rule

$\dots$ use C-R eqns in right spots
in the right way = $\bigcirc$

Exciting tool:



need f
analytic
l-1, onto
do right
thing on
boundary



$$u = y = \text{Im } z$$