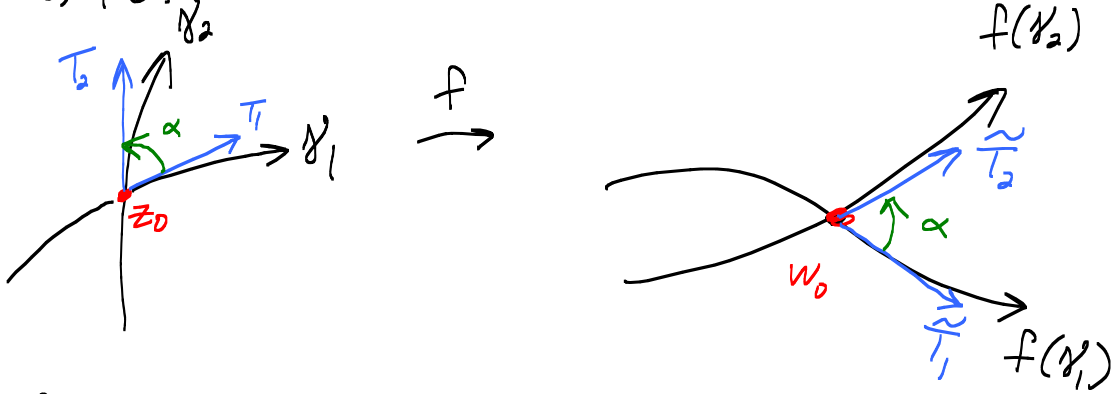
Lesson 32 Conformal maps

Exam 2 in class Wed., Nov 13
Practice problems on home page
plus old exam 2.

Big fact: If f is analytic and f' is non-vanishing, then f is conformal, meaning it preserves angles.

Why: Suppose $f'(z_0) \neq 0$. $f(z_0) = w_0$



Easy consequence of Chain rule #2:

$\gamma_j : z_j(t)$. $z_j(0) = z_0$ $j=1,2$

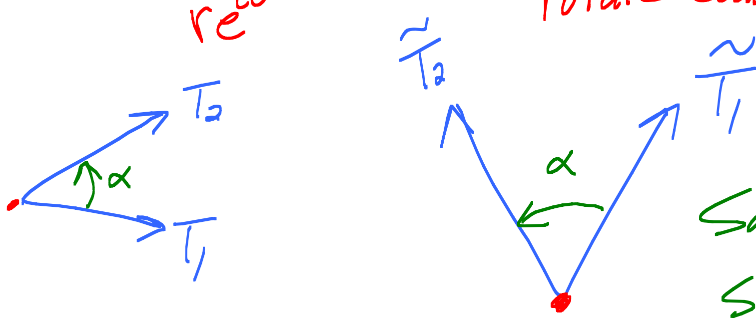
$T_j = z_j'(0)$

$f(\gamma_j) : f(z_j(t))$ $f(\underbrace{z_j(0)}_{z_0}) = w_0$

$\tilde{T}_j = \underbrace{f'(z_j(0))}\, z_j'(0)$

$f'(z_0)\, T_j$

Aha! $\tilde{T}_j = \underbrace{f'(z_0)}_{re^{i\theta}}\, T_j$ ← stretch by r, rotate counterclockwise by angle θ,



Same α!
Same "sense"

Cool fact: Reverse is true! (almost).

$$F: \mathbb{R}^2 \to \mathbb{R}^2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

Jacobian matrix: $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$

matrix mult is the linear map that determines how tangent vectors transform

Analytic geometry shows: linear map preserves angles only if it as a rotation map:

$$R \cdot \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

See C-R Equs!

Almost part: Need u, v C^1-smooth, $\det(\text{Jacobian}) \neq 0$.

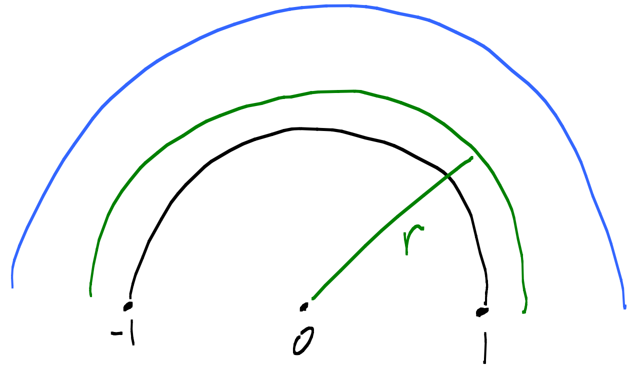
EX: $f(z) = z^2$. $f'(z) = 2z$. f preserves angles on $\mathbb{C} - \{0\}$.

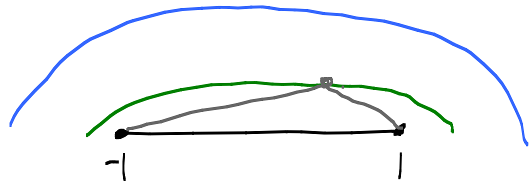
Doubles angles at $z=0$!

EX: Jukovsky map: $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

On circles: $\underset{\text{circle}}{J(re^{i\theta})} = \frac{1}{2}\left(r\underset{\cos\theta + i\sin\theta}{e^{i\theta}} + \frac{1}{r}\underset{\cos\theta - i\sin\theta}{e^{-i\theta}}\right)$

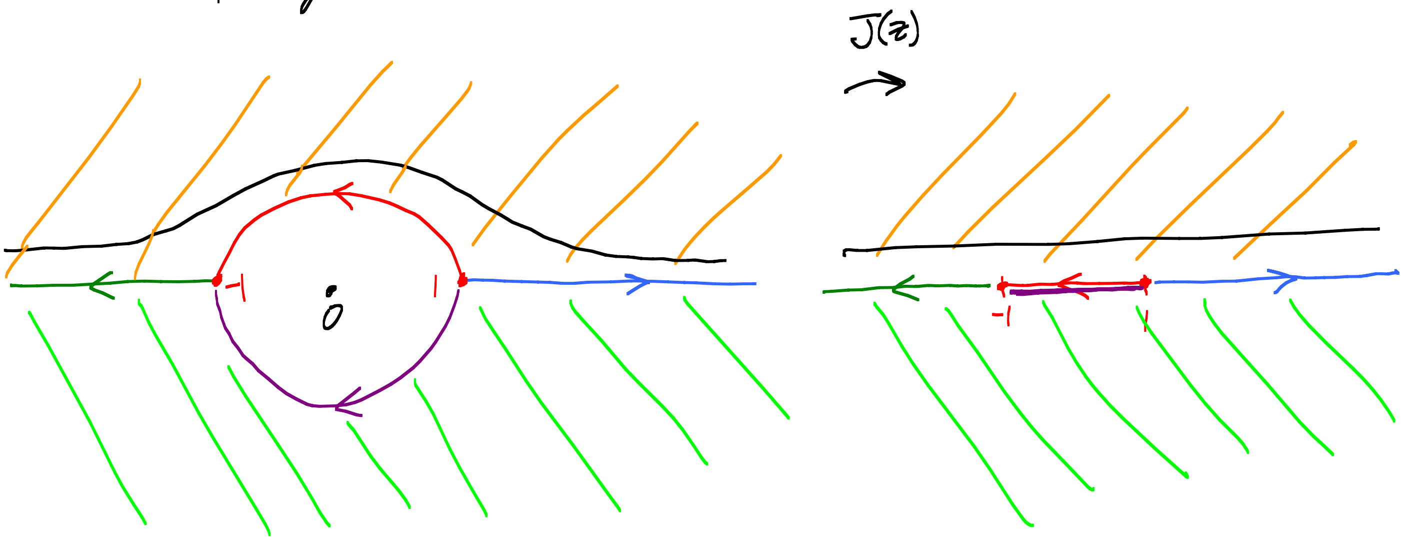
$$= \underbrace{\left[\tfrac{1}{2}\left(r + \tfrac{1}{r}\right)\right]\cos\theta + i\left[\tfrac{1}{2}\left(r - \tfrac{1}{r}\right)\right]\sin\theta}_{\text{ellipse}}$$

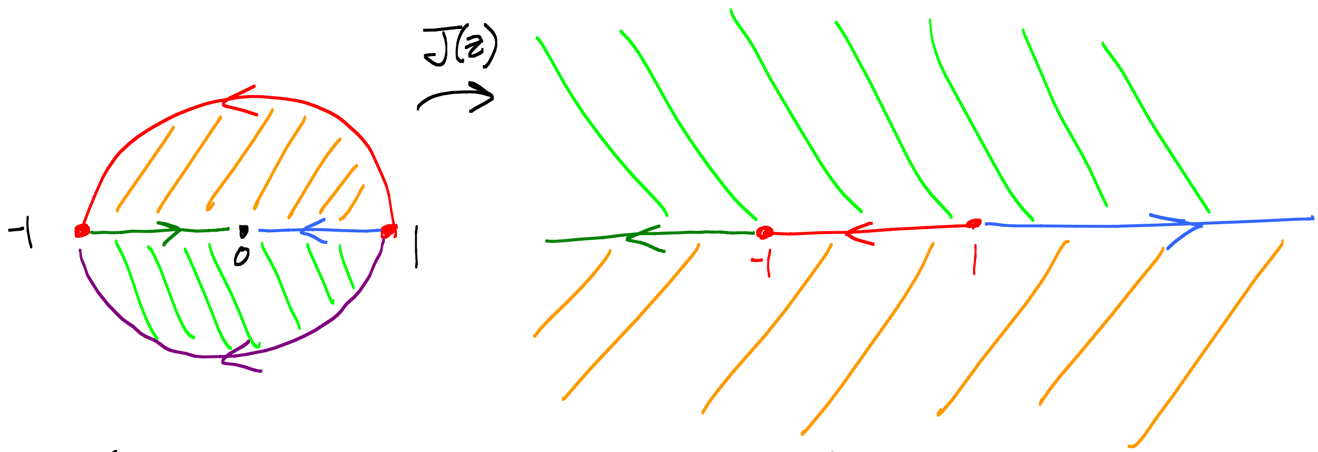




Foci at ± 1

High school analytic geom:

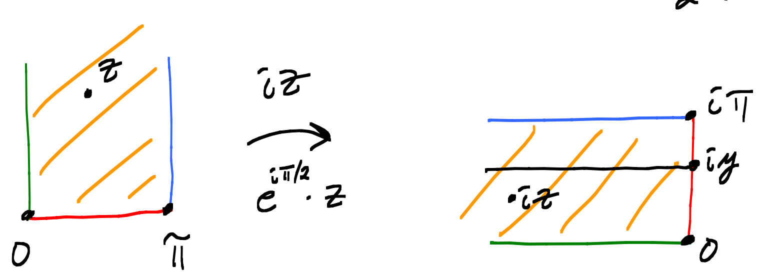




Complex cosine map:

$$\cos z = \tfrac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$= \tfrac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}}\right) = J\left(e^{iz}\right)$$

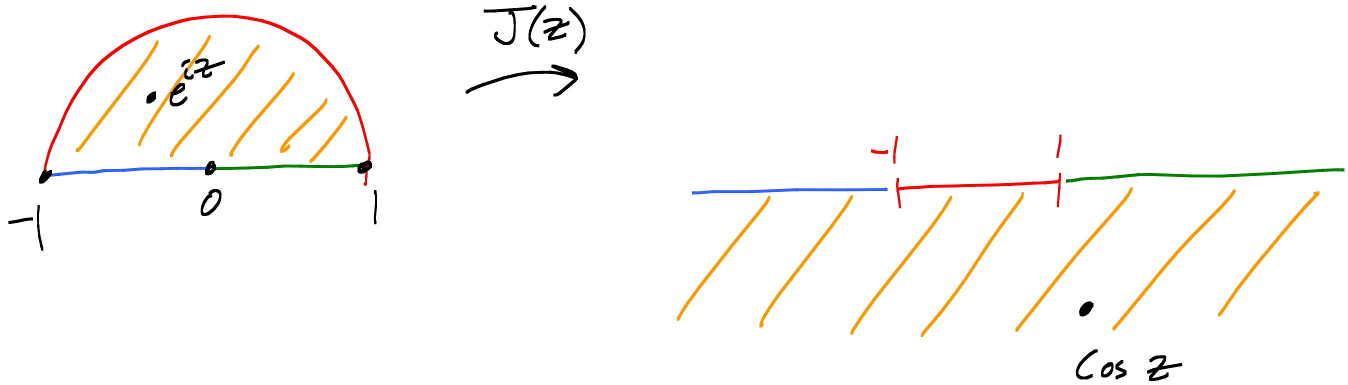


$e^z = e^x e^{iy}$

$\longrightarrow$

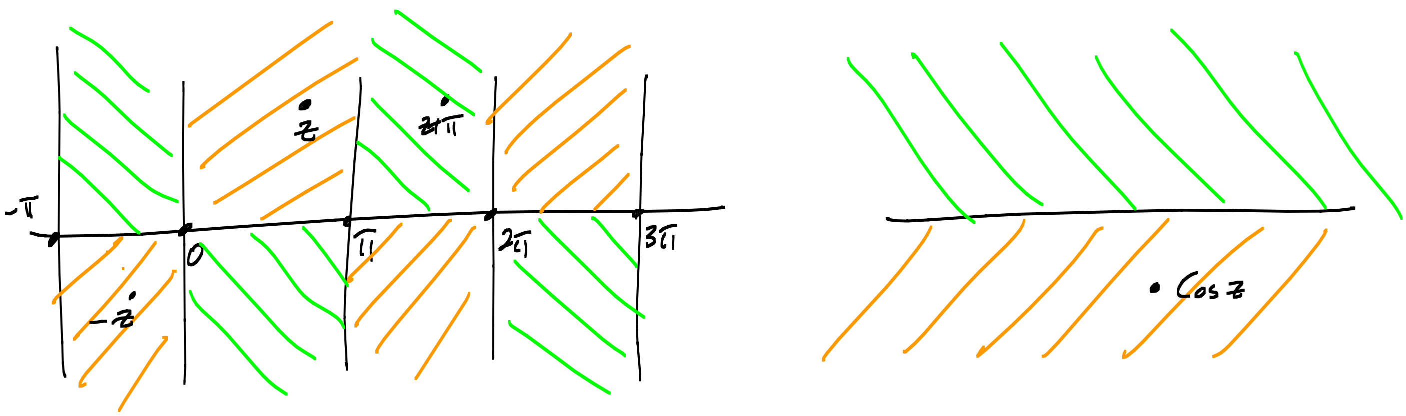
$-\infty < x < 0$

$0 < e^x < 1$

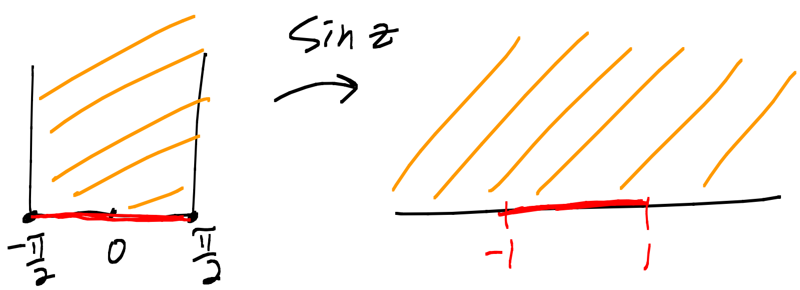


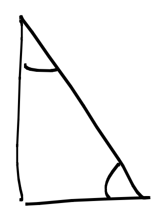
$\cos(-z) = \cos z$, $\quad \cos(z+\pi) = -\cos z$

$\cos(z+2\pi) = \cos z$



<u>What about $\sin(z)$?</u> $\quad \sin z = \cos\left(z - \frac{\pi}{2}\right)$





<u>Show</u>: $\cos \frac{1}{z}$ has an essential singularity at $z=0$.