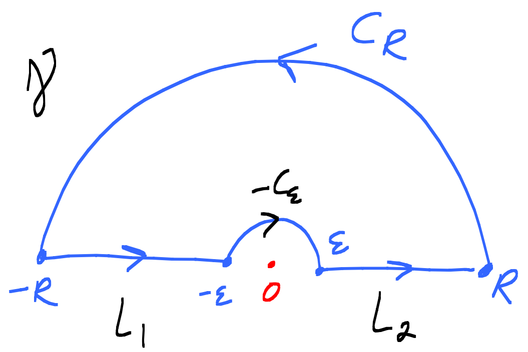
Lesson 32 Classic integrals

Exam 2 in class next Wed. (no HWK due)
Practice probs on home page. Review on Mon.



$$I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) = \operatorname{Re} \tfrac{1}{2}(1 - e^{i2x})$$

$$f(z) = \frac{\tfrac{1}{2}(1 - e^{i2z})}{z^2}$$

$$0 = \int_\gamma f\,dz = \underbrace{\left(\int_{L_1} + \int_{L_2}\right)}_{\text{want}} + \underbrace{\left(\int_{C_R}\right)}_{\substack{\to 0 \\ R\to\infty}} + \underbrace{\left(\int_{-C_\varepsilon}\right)}_{\substack{\to ? \\ \varepsilon\to 0}}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \left(\operatorname{Max}_{C_R} |f(z)|\right)\pi R \le \frac{\tfrac{1}{2}(1+1)}{R^2}\pi R$$

$|e^{i2z}| \le 1$
$\operatorname{Im} z \ge 0$

<u>Claim</u>: $\dfrac{\tfrac{1}{2}(1-e^{i2z})}{z^2}$ has a <u>simple pole</u> at $z = 0$

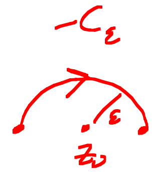
$$\frac{\tfrac{1}{2}\left[1 - \left(1 + (i2z) + \frac{(i2z)^2}{2!} + \cdots\right)\right]}{z^2} = \frac{-iz + z^2 + \cdots}{z^2}$$

$$= -\frac{i}{z} + \underbrace{1 + \text{regular power series}}_{H(z)}$$

So
$$\int_{-C_\varepsilon} f(z)\,dz = \underbrace{\int_{-C_\varepsilon} -\frac{i}{z}\,dz}_{-\int_0^\pi \frac{-i}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt} + \underbrace{\int_{-C_\varepsilon} H\,dz}_{\substack{\to 0 \\ \text{as } \varepsilon \to 0}}$$

$\to -\pi$ as $\varepsilon \to 0$

Fact: At a simple pole of f



$$\lim_{\varepsilon\to 0} \int_{-C_\varepsilon} f\, dz = \underset{\substack{\uparrow \\ \text{because} \\ \text{clockwise}}}{-}\ \underset{\substack{\uparrow \\ \text{only half the circle}}}{\frac{1}{2}} (2\pi i)\, \operatorname{Res}_{z_0} f$$

Note: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$ is absolutely integral. So $\displaystyle\underbrace{\lim_{\substack{\varepsilon\to 0\\ R\to\infty}}}_{\text{P.V.}} \int = \int_{-\infty}^{\infty}$

Hated (stupid) formula: f has a pole of order N at z_0.

$$f(z) = \frac{A_N}{(z-z_0)^N} + \cdots + \frac{\overset{\swarrow \text{Res}}{A_1}}{z-z_0} + \text{power series}$$

$$\underbrace{(z-z_0)^N f(z)}_{F(z) \text{ has a removable sing at } z_0} = A_N + \cdots + A_1 (z-z_0)^{N-1} + \text{pow. ser.}$$

$$A_1 = \operatorname{Res}_{z_0} f = \frac{\left.\frac{d^{N-1}}{dz^{N-1}} F(z)\right|_{z_0}}{(N-1)!} = \underbrace{\lim_{z\to z_0} \frac{\frac{d^N}{dz^{N-1}}\left[(z-z_0)^N f(z)\right]}{(N-1)!}}_{\text{True, but dumb!}}$$

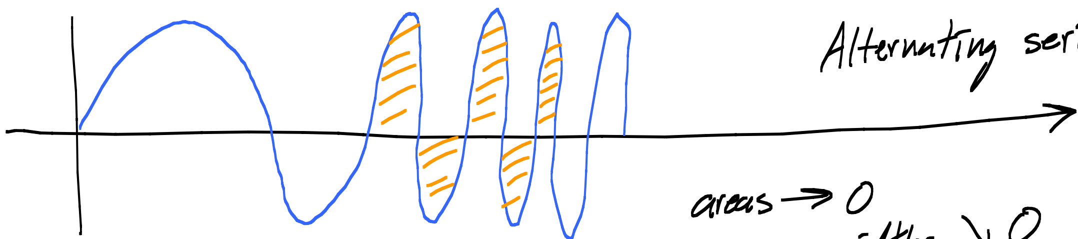
EX: Better way. $\dfrac{\operatorname{Log} z}{(z^2+1)^4}$ has a pole of order 4 at $z=i$

$$\frac{\operatorname{Log} z}{(z^2+1)^4} = \frac{\operatorname{Log} z}{[(z-i)(z+i)]^4} = \frac{1}{(z-i)^4}\left[\underbrace{\frac{\operatorname{Log} z}{(z+i)^4}}_{B_0 + B_1(z-i) + B_2(z-i)^2 + B_3(z-i)^3 + \cdots}\right]$$

$$= \frac{B_0}{(z-i)^4} + \cdots + \frac{B_3}{z-i} + \text{p.s.}$$

$$\text{So} \quad \text{Res}_i = B_3 = \frac{\frac{d^3}{dz^3}\left[\frac{\text{Log}\, z}{(z+i)^4}\right]\Big|_{z=i}}{3!}$$

Classic integrals: Fresnel integrals $\displaystyle\int_0^\infty \sin x^2\, dx = \lim_{R\to\infty} \int_0^R$



Alternating series test shows lim exists.

areas $\to 0$ because widths $\searrow 0$.

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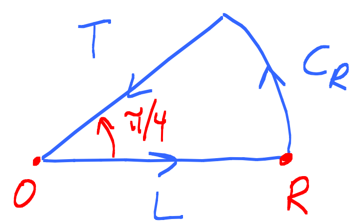


$f(z) = e^{-z^2}$

$$\int_\gamma f\, dz = 0$$

$$\int_L f\, dz = \int_0^R e^{-x^2}\, dx \to \frac{\sqrt{\pi}}{2}$$

as $R \to \infty$

$$\left(\int_0^\infty e^{-x^2} dx\right)^2$$

$$= \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$-T$: $z(t) = t\, e^{i\pi/4}$, $0 \le t \le R$

$$\int_T f\, dz = -\int_{-T} f\, dz = -\int_0^R e^{-(te^{i\pi/4})^2} \underbrace{\left[e^{i\pi/4}\, dt\right]}_{dz}$$

$$= -\int_0^R e^{-it^2}\, dt = -\left(\int_0^R \cos t^2\, dt - i\int_0^R \sin t^2\, dt\right) e^{i\pi/4}$$

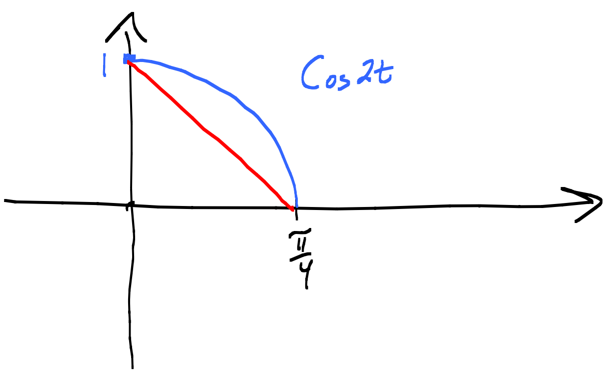
Last step: $\left|\int_{C_R} f\,dz\right| = \left|\int_0^{\pi/4} e^{-(Re^{it})^2} \underbrace{\left[iRe^{it}\,dt\right]}_{dz}\right|$

Basic estimate fails us!

$$\leq \int_0^{\pi/4} \left|e^{-R^2 e^{i2t}}\right| R\,dt$$

$$\leq \int_0^{\pi/4} R\, e^{-R^2 \cos 2t}\,dt$$

Think "Jordan's Lemma"



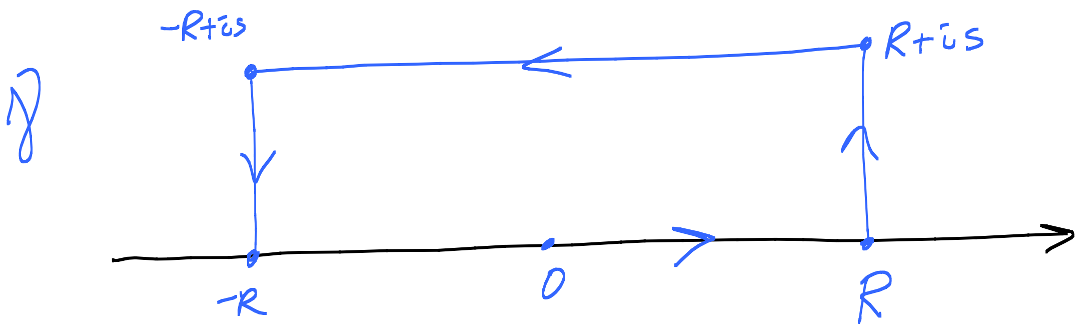
$$e^{-R^2 \cos 2t} \leq e^{-R^2\left(1-\frac{4}{\pi}t\right)}$$

easy to integrate

Show $\int_{C_R} \to 0$

Classic integral: $e^{-\pi s^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi xs}\,dx$

Fourier transform of $e^{-\pi x^2}$ is itself!



$f(z) = e^{-\pi z^2}$

Int along bottom: $\int_{-\infty}^{\infty} e^{-\pi t^2}\,dt$ known.

Int along top $= -\widetilde{\mathcal{F}}\left[e^{-\pi x^2}\right](s)\; e^{\pi s^2}$

Show Int over sides $\longrightarrow 0$ as $R \to \infty$.

Practice problem: $\quad \mathrm{Res}_{z_0} \dfrac{f(z)}{g(z)}$ $\quad$ f, g analytic near z_0, g has a double zero at z_0.

$g(z_0) = 0$
$g'(z_0) = 0$
$g''(z_0) \neq 0$

Hint: $g(z) = a_2(z-z_0)^2 + \cdots$

$$= (z-z_0)^2 \underbrace{\left[a_2 + a_3(z-z_0) + \cdots\right]}_{G(z)}$$