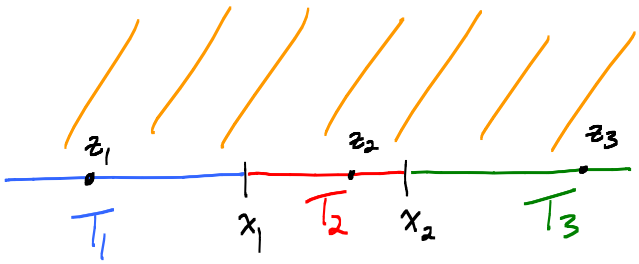
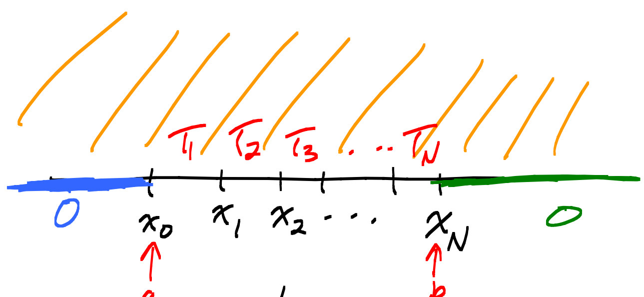
Lesson 39 More heat problems

Last HWK due Wed., Dec. 4



$$u(z) = \underbrace{A}_{A=\frac{T_1-T_2}{\pi}} \operatorname{Arg}(z-x_1) + \underbrace{B}_{B=\frac{T_2-T_3}{\pi}} \operatorname{Arg}(z-x_2) + \underbrace{C}_{C=T_3}$$

Cool thing!



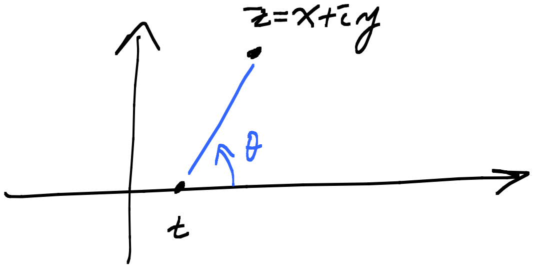
$$u(z) = \frac{1}{\pi}\sum_{1}^{N} (-\Delta T) \operatorname{Arg}(z-z_k)$$ ← A Riemann sum!

$$\longrightarrow -\frac{1}{\pi}\int_a^b T'(t) \operatorname{Arg}(z-t)\,dt$$ $\Delta T \approx \frac{dT}{dt}\Delta t$

Solves the Dirichlet problem on the UHP given data $T(x)$ on $\mathbb{R}$ that is C^1 smooth and supported in $[a,b]$.

But there's more:

$$u(z) = -\frac{1}{\pi}\int_{\mathbb{R}} \underbrace{\operatorname{Arg}(z-t)}_{u} \underbrace{T'(t)\,dt}_{dv} = \frac{1}{\pi}\int_{\mathbb{R}} T(t) \frac{d}{dt}\operatorname{Arg}(z-t)\,dt$$



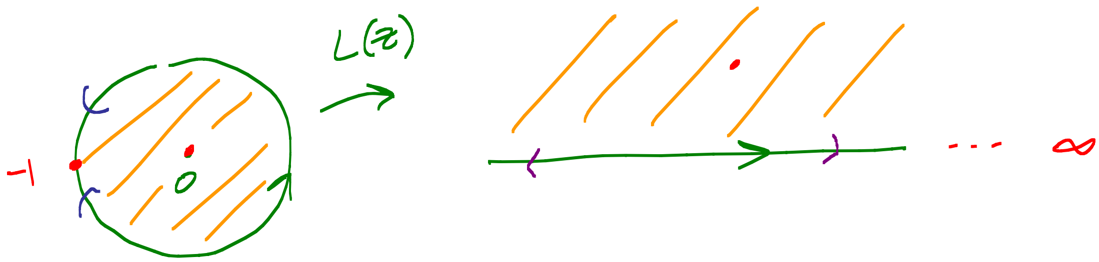
$$\operatorname{Arg}(z-t) = \operatorname{Tan}^{-1} \frac{y}{x-t}$$

So
$$\frac{d}{dt} \operatorname{Tan}^{-1} \frac{y}{x-t} = \frac{1}{1+\left(\frac{y}{x-t}\right)^2} \left(\frac{y}{(x-t)^2}\right)$$
$$= \frac{y}{(x-t)^2+y^2}$$

Poisson's formula for UHP!

$$u(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} T(t) \frac{y}{(x-t)^2+y^2}\, dt$$

Big idea: Use a LFT to pull back formula to $D_1(0)$.



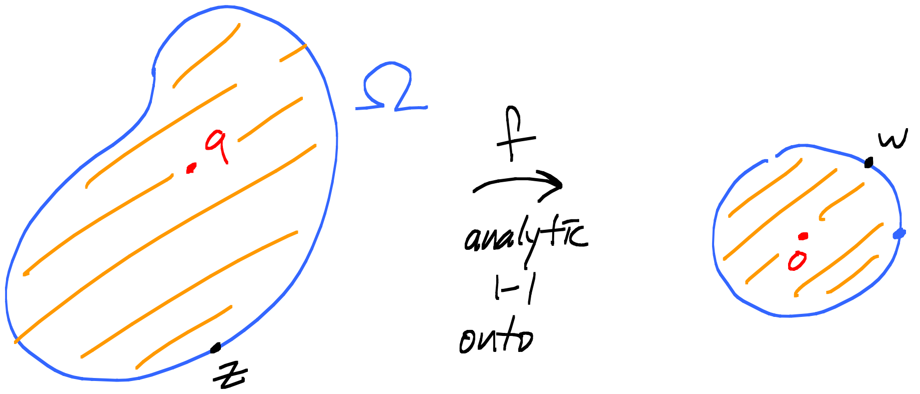
Get Poisson kernel for $D_1(0)$

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} T(e^{i\theta})\, d\theta$$

See: this solves D. Prob: Find u continuous on $\overline{D_1(0)}$,
harmonic on $D_1(0)$,
$u(e^{i\theta}) = $ given $T(e^{i\theta})$.

The power of the Riemann mapping theorem

Ω no holes.
$\Omega \neq \mathbb{C}$ (Liouville!)



f "Riemann map"

Carathéodory's Thm: Boundary of Ω a Jordan curve (simple, closed, cont.), then f is continuous up to the boundary.

1. Given temp on bndry of Ω: $T(z)$

2. Get temp fcn on $C_1(0)$: $\underline{T(f^{-1}(w))}$

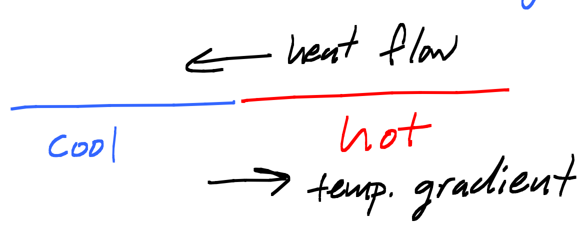
3. Solve heat prob on $D_1(0)$ via Poisson's: Get $u(w)$.

4. Sol^n on Ω: $U(z) = u(f(z)) \leftarrow$ harmonic ✓

On bndry: $U(z) = u(\underbrace{f(z)}_{w}) = T(f^{-1}(\underbrace{f(z)}_{w})) = T(z)$ ✓

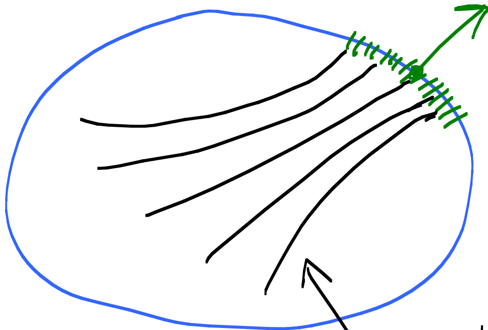
Lesson 39: Insulated parts of boundary.

Kelvin's Law: (Rate of heat flow) is proportional to the (temp. gradient)



constant is < 0 !

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n} = 0$$ ← isotherms $\perp$ to boundary

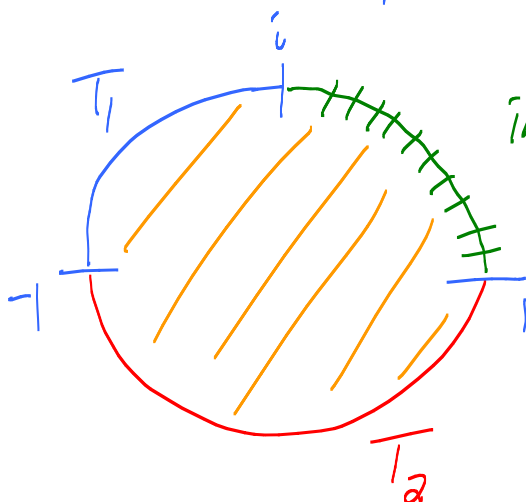


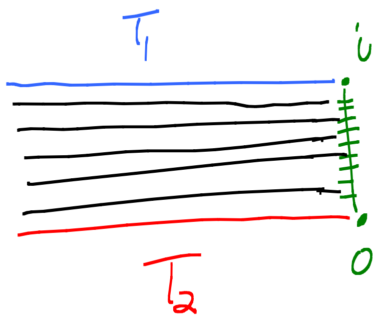
↖ insulated: no heat flow!

isotherms: curves of const temp $u=c$.

Aha! Conformal maps preserve this $\perp$ ity.

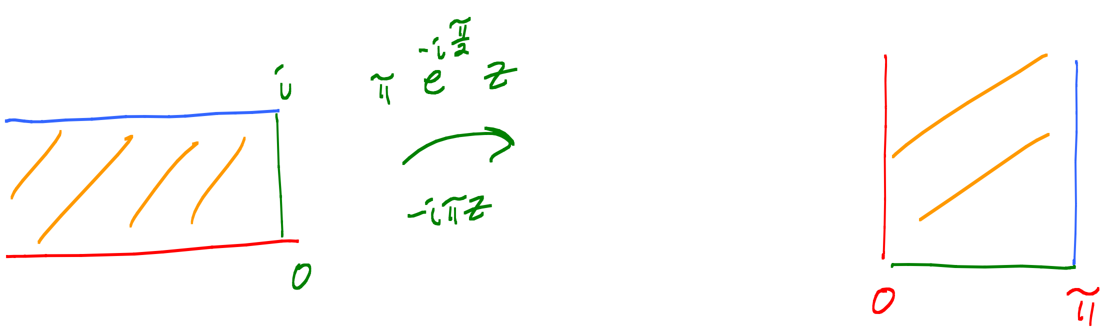
Prob:

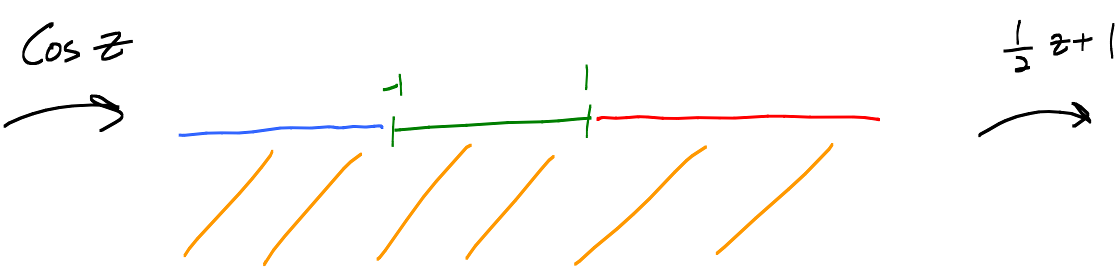


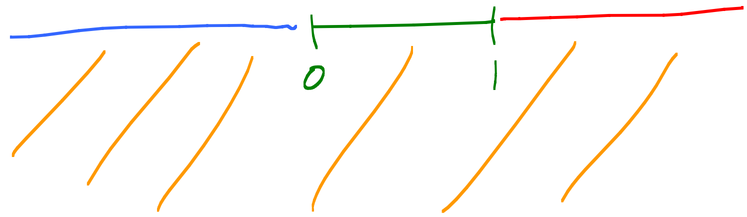


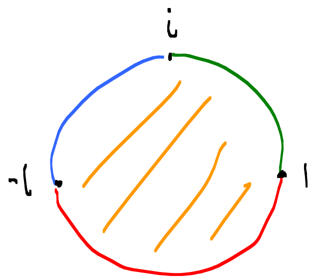
$$u(z) = A \underbrace{\operatorname{Im} z}_{y} + B$$

$$= (T_1 - T_2)\, y + T_2$$









$$L(z) = \frac{1+i}{1-i} \cdot \frac{z-i}{z+1}$$

Sol^{n}: $u\left(\frac{1}{-i\pi} \cos^{-1}\left(2L^{-1}(z) - 1 \right) \right)$