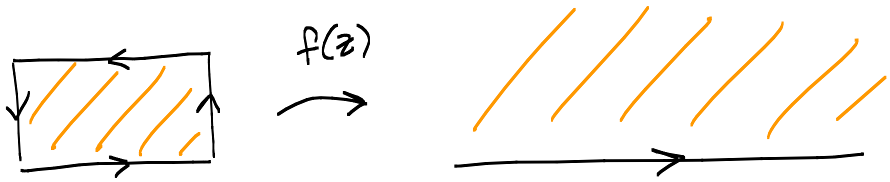
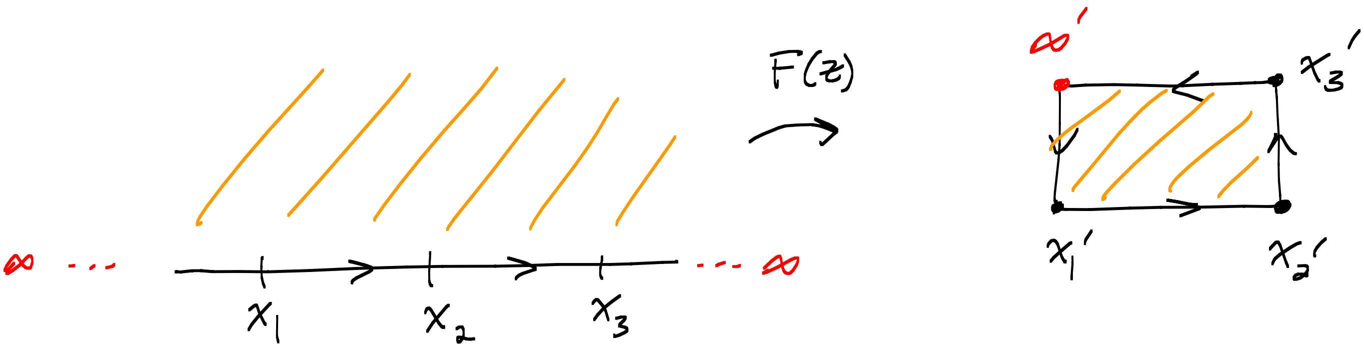
Lesson 37 Schwarz–Christoffel maps

Hwk 12: 36, 38, 39 due Wed, Dec 4

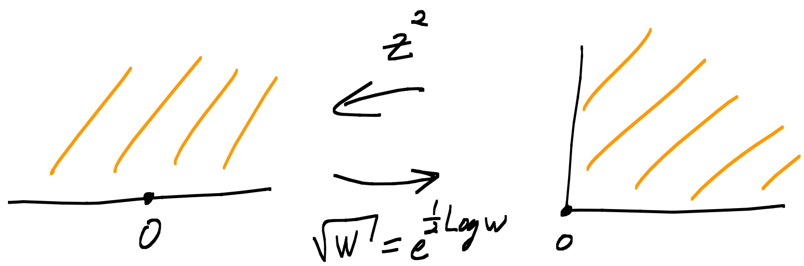
Prob:



Easier:



Hmmm.

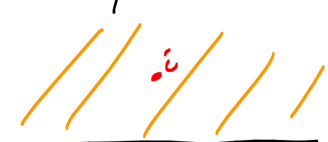


Note that $\frac{d}{dw}\left(w^{1/2}\right) = \frac{1}{2} w^{-1/2}$

Big idea: Try to find a map such that

$$F'(z) = \underbrace{A\,(z-x_1)^{-1/2}(z-x_2)^{-1/2}(z-x_3)^{-1/2}}_{H(z),\ \text{analytic on UHP}}$$

Analytic $F(z)$ exists and is computable:

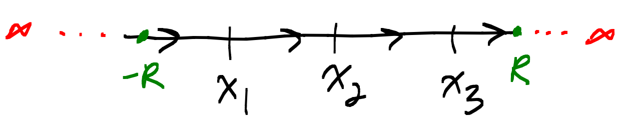


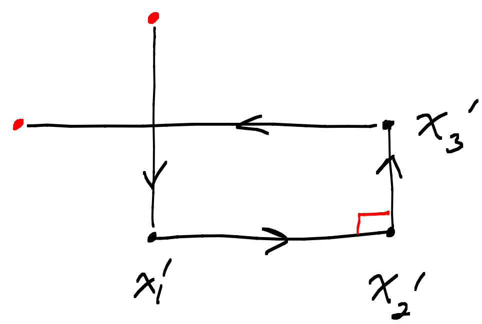
$$F(z) = \int_{L_i^z} H(w)\,dw$$

It works! Why: On $\mathbb{R}$:

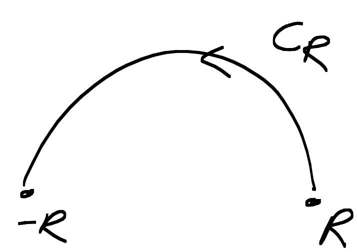
$$F'(t) = A(t-x_1)^{-1/2}(t-x_2)^{-1/2}(t-x_3)^{-1/2}$$

Passing x_j leads to a change of sign under the square root – kicking in a multiplication by i, meaning tangent vector jumps $90°$.





Amazing luck. Red dots hook up!



Why: $\left|F(-R)-F(R)\right| = \left|\int_{C_R} F'\,dz\right|$

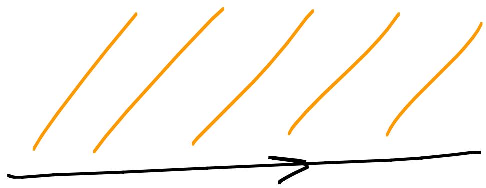
$$\le \left(\max_{C_R}|F'|\right)\pi R$$

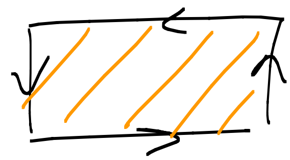
and $\left|F'(z)\right| = \dfrac{A}{\sqrt{\underbrace{|z-x_1||z-x_2||z-x_3|}_{\text{Poly of deg } 3 \,\ge\, a|z|^3 \text{ if } |z|>\tilde{R}.}}}$

$$\le \frac{A}{\sqrt{aR^3}} \quad \text{if} \quad |z| = R > \tilde{R}$$

So $\left|F(-R) - F(R)\right| \le \dfrac{A}{\sqrt{a}} \, \dfrac{\pi R}{R^{3/2}} \to 0$ as $R \to \infty$.

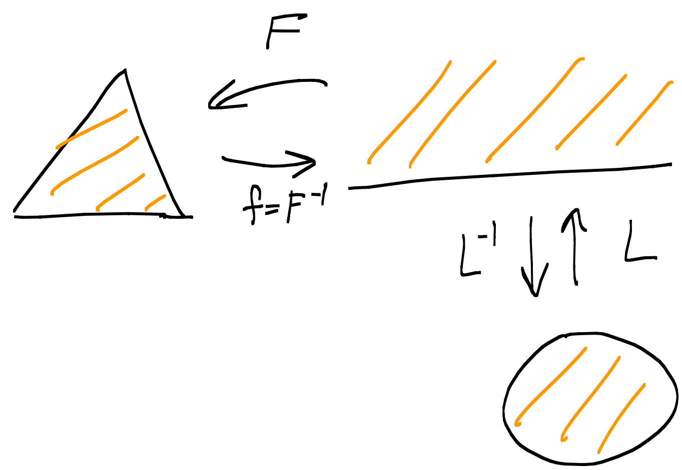
Red dots hook up!



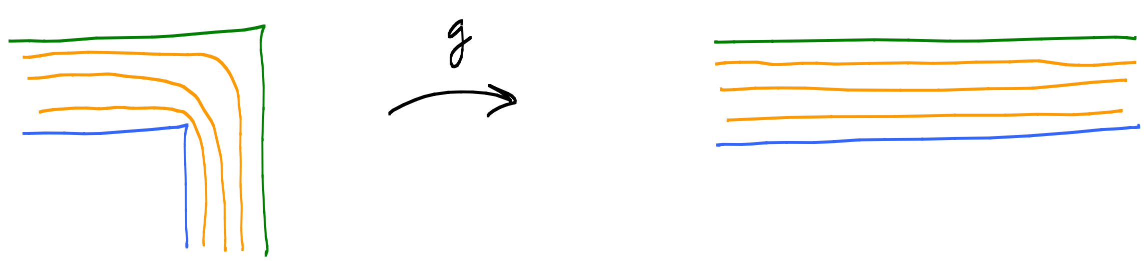


Last step: Argument principle shows F is 1-1 onto the rectangle!

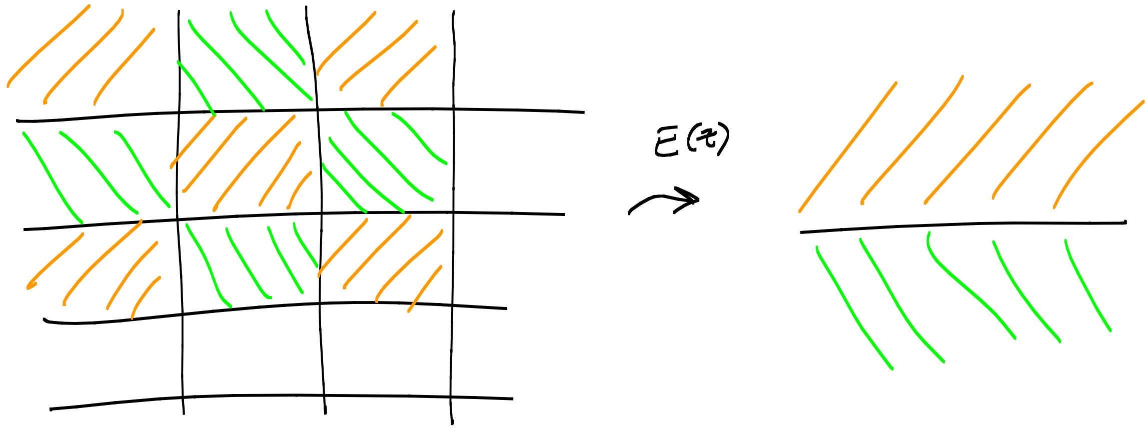
Famous S-C maps:



Similarly for any polygon.



Cool thing:



Schwarz-reflection yields a doubly periodic fcn.

Elliptic fcn :
$$E(z+a) = E(z)$$
$$E(z+ib) = E(z)$$