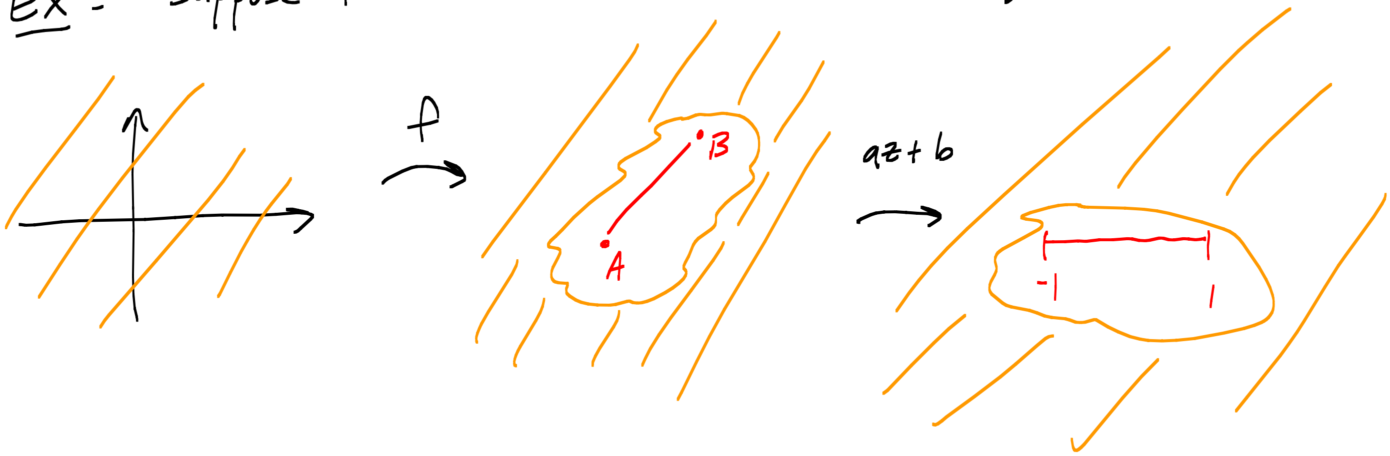
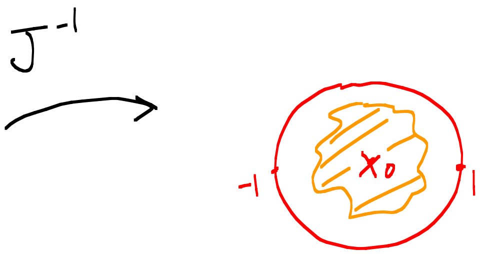
Lesson 40 Schwarz lemma

Practice problems for final on home page

How big a set can an entire fcn miss?

EX: Suppose f entire and misses a line segment



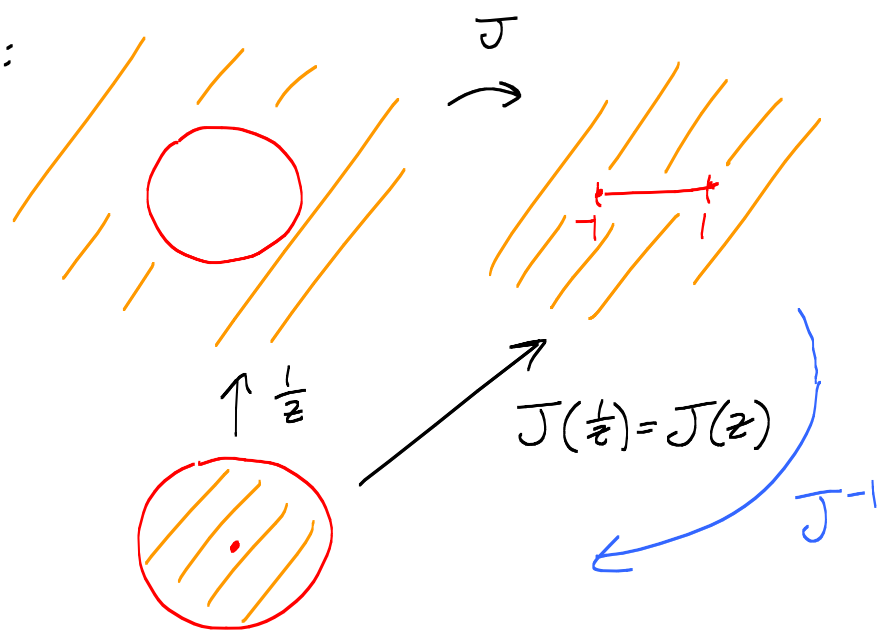


Get bndd entire fcn!

Liouville's $\Rightarrow$ const.

$\Rightarrow$ $f \equiv$ const.

Note:



Picard's Theorem: An entire fcn that misses two points must be constant!

EX: e^z misses 0. Every other point gets hit ∞ many times.

Schwarz lemma: Suppose f is analytic on $D_1(0)$ and $f: D_1(0) \to D_1(0)$ with $f(0)=0$.

Then (A) $|f(z)| \le |z|$ for all z, and

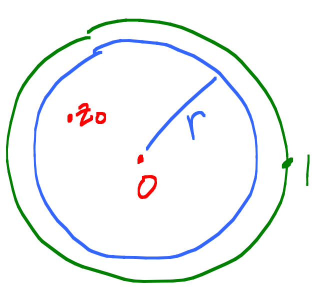
(B) $|f'(0)| \le 1$

Furthermore, if equality holds in (A) for some $z_0 \ne 0$, or if equality holds in (B), then f must be a rotation: $f(z) = \lambda z$, $\lambda = e^{i\theta}$ (a const of modulus 1)

Why:
$$F(z) = \begin{cases} \dfrac{f(z)}{z} & z \ne 0 \\ f'(0) & z = 0 \end{cases}$$

$$DQ = \frac{f(z) - \underbrace{f(0)}_{=0}}{z - 0} = \frac{f(z)}{z}$$

$F(z) \to F(0)$ as $z \to 0$. So $z=0$ is a removable sing for F.



$C_r(0)$

Pick $z_0 \in D_1(0)$ and choose r with $0 \le |z_0| < r < 1$. Max princ shows that

$$|F(z_0)| \le \max_{|z|=r} |F(z)| = \max_{|z|=r} \frac{|f(z)|}{r} \le \frac{1}{r}$$

Aha! Can let $r \nearrow 1$. Then $\frac{1}{r} \searrow 1$. Conclude that $|F(z_0)| \le 1$. Since z_0 was arbitrary, we see that $|F(z)| \le 1$ for all z. Get (A) and (B)!

Furthermore part follows from: Max princ: An analytic fcn that assumes a local max inside domain must be const. [If $|f'(0)| = 1$, then $|F(z)|$ assumes max at $z=0$. So $F(z) \equiv \lambda$. $F(0) = f'(0)$ has modulus one. So $|\lambda| = 1$. Similarly for (A) part.]

Cor: If f is analytic on $D_1(0)$ and maps $D_1(0)$ one-to-one onto itself with $f(0)=0$, then $f(z) \equiv \lambda z$ with $|\lambda| = 1$.

Why: Schwarz $\Rightarrow |f'(0)| \le 1$.

f^{-1} also satisfies hyp. So $\underbrace{|(f^{-1})'(0)|}_{\frac{1}{|f'(0)|}} \le 1$.

$\Rightarrow |f'(0)| \ge 1$ too.

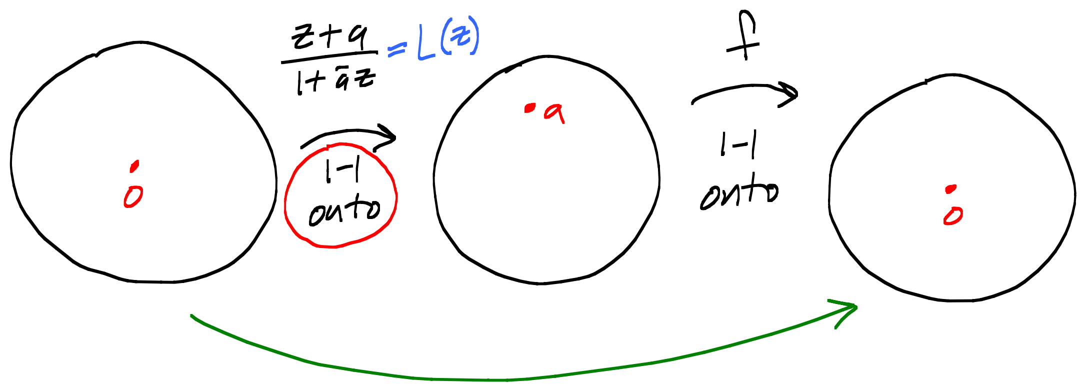
So $|f'(0)| = 1$ and Schwarz $\Rightarrow f(z) \equiv \lambda z$, $|\lambda| = 1$.

Möbius transformations: $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$ where $a \in D_1(0)$.

Big fact: If f is analytic on $D_1(0)$ and maps $D_1(0)$ one-to-one onto itself, then $f(z) = \lambda\,\varphi_a(z)$ for some $a \in D_1(0)$ and λ with $|\lambda| = 1$.

Automorphism group of the unit disc $= Aut(D_1(0))$ is the set of all such maps.

Why:



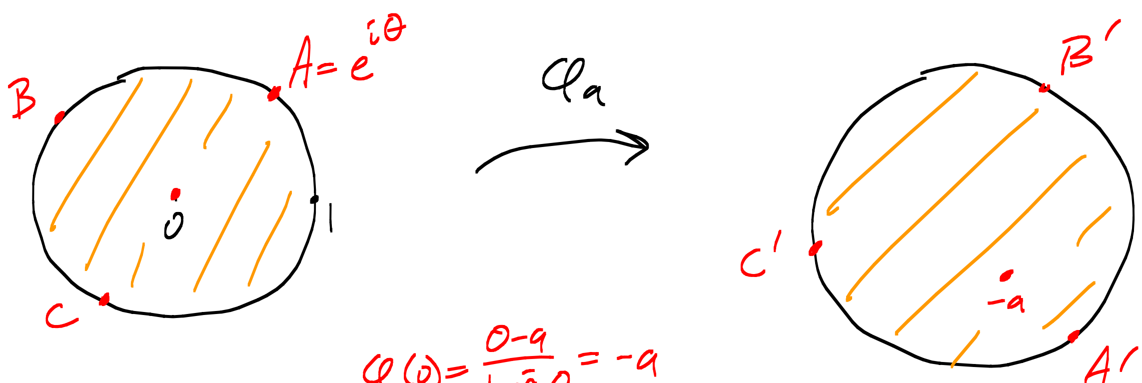
$F(z)$, $F(0)=0$.

Cor: $F(z) = \lambda z$

$f(\underbrace{L(z)}_{w}) = \lambda z$ $\quad z = L^{-1}(w) = \frac{w-a}{1-\bar{a}w}$

So $f(w) = \lambda \frac{w-a}{1-\bar{a}w}$ ✓

Note: Why are Möbius transf 1-1 onto maps of $D_1(0)$ ⟲ ?



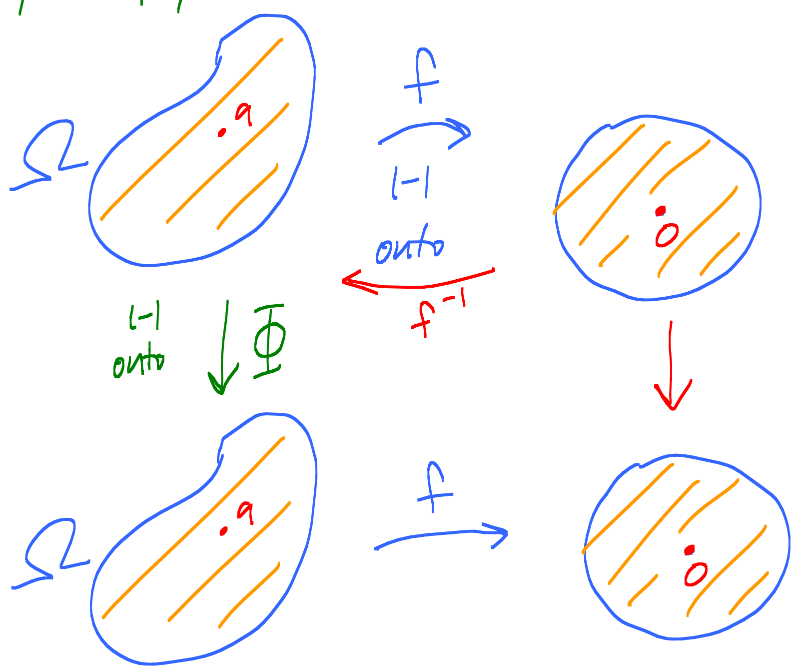
$\varphi_a(0) = \frac{0-a}{1-\bar{a}\cdot 0} = -a$

"one point test"

A: $\dfrac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}} \cdot \dfrac{e^{-i\theta}}{e^{-i\theta}} = \underbrace{\dfrac{e^{i\theta}-a}{e^{-i\theta}-\bar{a}}}_{\frac{w}{\bar{w}}} \cdot e^{-i\theta}$

$\frac{w}{\bar{w}}$ ← unimodular!

Riemann mapping theorem shows how to get "automorphisms" of any simply connected domain $\Omega \neq \mathbb{C}$:



$f \circ \Phi \circ f^{-1} = \lambda \varphi_a$

Get a formula for Φ:

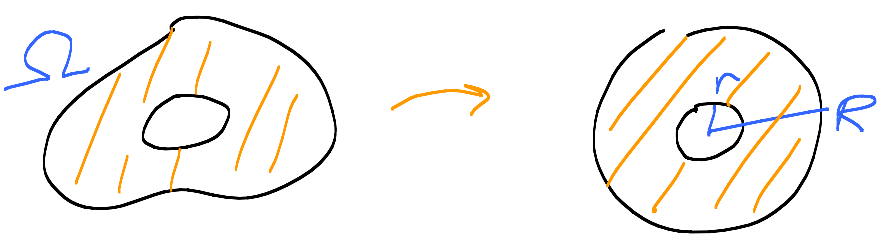
$f(\Phi(\underbrace{f^{-1}(z)}_{w})) = \lambda \varphi_a(z)$

$z = f(w)$

$f(\Phi(w)) = \lambda \varphi_a(f(w))$

$\Phi(w) = f^{-1}(\lambda \varphi_a(f(w)))$

(Conformal maps are <u>unique</u> up to automorphisms.)



$\frac{r}{R}$ depends on Ω