Review for Exam 1

Exam 1 in class Monday

1. Show that if $\varphi(x,y)$ is a twice continuously differentiable real valued *harmonic* function on a domain, then

$$\underbrace{\frac{\partial\varphi}{\partial x}}_{u} - i\,\underbrace{\frac{\partial\varphi}{\partial y}}_{v = -\frac{\partial\varphi}{\partial y}}$$

$u = \frac{\partial\varphi}{\partial x}$

is analytic there.

$\mathcal{C}^1$-smooth! Important when applying C-R Equs.

1. $$u_x \overset{?}{=} v_y$$

$$\frac{\partial}{\partial x}(\varphi_x) \overset{?}{=} \frac{\partial}{\partial y}\left(-\frac{\partial\varphi}{\partial y}\right)$$

$$\varphi_{xx} = -\varphi_{yy}$$ Yes! $\Delta\varphi \equiv 0.$

2. $$u_y \overset{?}{=} -v_x$$

$$\frac{\partial}{\partial y}(\varphi_x) \overset{?}{=} -\frac{\partial}{\partial x}(-\varphi_y)$$

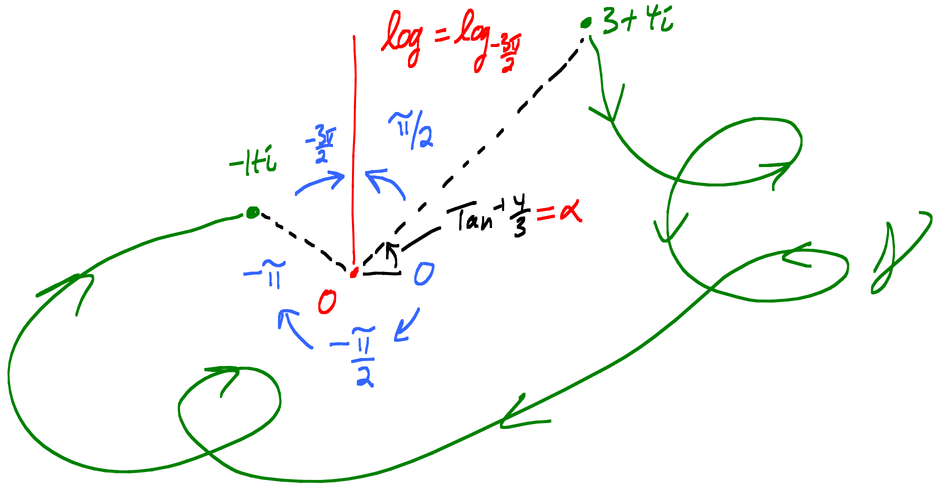
$$\varphi_{yx} = \varphi_{xy}$$ ← yes, when φ is $\mathcal{C}^2$-smooth ✓

$\mathcal{C}^1$-smooth u, v plus C-R Equs $\Longrightarrow$ fcn analytic

3. Compute

$$\int_\gamma \frac{1}{z}\,dz$$

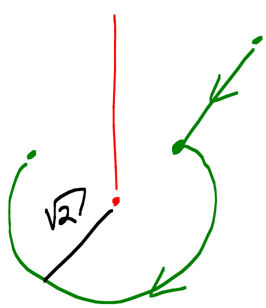
where γ is any curve in the plane that starts at $3+4i$ and ends at $-1+i$ and that avoids the set $\{z : z = it, t \geq 0\}$ (i.e., the positive imaginary axis, including $z = 0$).



$$\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}(\log z)\,dz = \log z \Big|_{START}^{END}$$

$$= \log\left(\sqrt{2}\,e^{-\frac{5\pi}{4}i}\right) - \log\left(5e^{i\alpha}\right)$$

$$= \left(\ln\sqrt{2} - i\frac{5\pi}{4}\right) - \left(\ln 5 + i\,\mathrm{Tan}^{-1}\frac{4}{3}\right)$$



Since $\frac{1}{z} = \frac{d}{dz}(\log z)$, $\int$ is independent of path. So pick a nice one.

4. Define

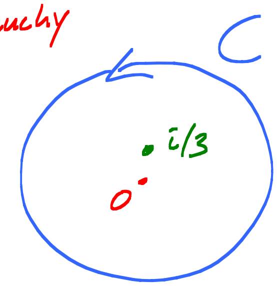
$$I(a) = \int_C \frac{e^{5iz}}{(z-a)^4}\, dz,$$

$4 = 3 + 1 \quad n = 3$ $\quad \cdot\, 3i$

where C is the unit circle parameterized in the counter clockwise direction and a is a complex number not on the unit circle. Compute $I(\frac{i}{3})$ and $\underbrace{I(3i)}_{=0}$. Is $I(a)$ an analytic function of a on the unit disc? Explain.

by Cauchy

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$



$$\frac{3!}{2\pi i} I(a) = f^{(3)}(a) \quad \text{of} \quad f(z) = e^{i5z}.$$

$$f' = i5e^{i5z}$$

$$f'' = (i5)^2 e^{i5z} = -25e^{i5z}$$

$$f''' = -i\,125 e^{i5z}$$

So $\quad I(a) = \dfrac{2\pi i}{6}\left(-i\,125\, e^{i5a}\right) \leftarrow$ analytic in a for a in $D_1(0)$ ✓

5. Show that

$$|e^{(z^2)}| \le e^{|z^2|}$$

and identify conditions for equality to hold. Is it true that $|e^{(z^2)}|$ tends to infinity as $|z|$ tends to infinity?

$$z = x + iy$$

$$z^2 = (x^2 - y^2) + i\,2xy$$

$$|e^{z^2}| = \left|e^{(x^2-y^2)+i\,2xy}\right| = \left|e^{x^2-y^2}\cdot e^{i2xy}\right|$$

$$= \underbrace{\left|e^{x^2-y^2}\right|}_{e^{x^2-y^2}} \cdot \underbrace{\left|e^{i2xy}\right|}_{=1}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$= e^{x^2-y^2}$$

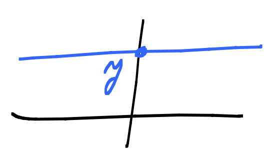
$$\overset{?}{\le} e^{|z^2|} = e^{|z|^2} = e^{x^2+y^2}$$

Yes!

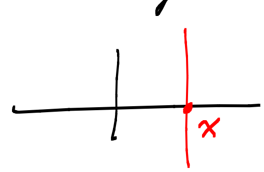
$$x^2 - y^2 \le x^2 + y^2 \leftarrow = \text{ only when } y = 0$$

$$e^{x^2-y^2} \le e^{x^2+y^2} \leftarrow = \text{ only when } y = 0$$

Funny thing: $|e^{z^2}| = e^{x^2-y^2}$ $\longrightarrow \infty$ when $|x| \to \infty$, y fixed



$\longrightarrow 0$ when $|y| \to \infty$, x fixed



5. Show that ~~an~~ a real valued analytic function on a domain ~~that has constant modulus~~ must be constant.

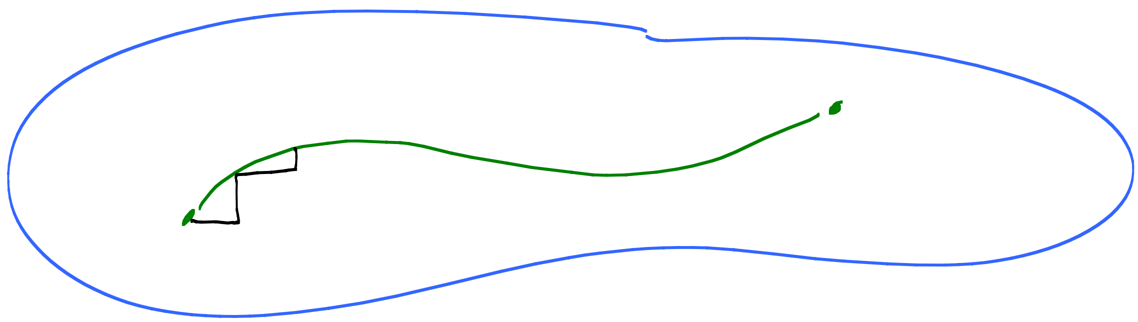
$$f(x+iy) = u(x,y) + i \underbrace{v(x,y)}$$

f real valued means $v \equiv 0$.

u,v have first partials.

Analytic $\Rightarrow$ C-R Eqns: $\begin{cases} u_x = v_y \equiv 0 \\ u_y = -v_x \equiv 0 \end{cases}$

$\nabla u \equiv 0. \Rightarrow$ u constant on the domain.



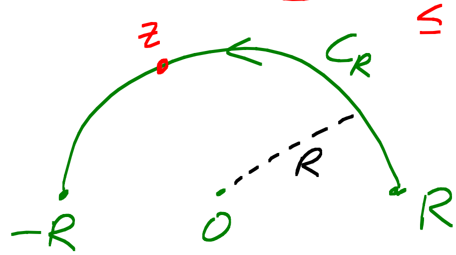
u const, $v \equiv 0$. So f is const.

12. Let C_R denote the half circle parameterized by $z(t) = Re^{it}$ for $0 \le t \le \pi$. Show that

$$|I| = \left| \int_{C_R} \frac{1}{z^4+1}\, dz \right| \le \underbrace{\left(\operatorname*{Max}_{C_R} \left| \frac{1}{z^4+1} \right| \right)}_{\le \frac{1}{R^4-1}} \underbrace{\text{Length}(C_R)}_{\pi R} \le \frac{1}{R^4-1} \cdot \pi R$$

when $R > 1$

tends to zero as R tends to infinity.



Denominator estimate: $\quad |a - b| \ge \big| |a| - |b| \big|$, $\quad b = -B$

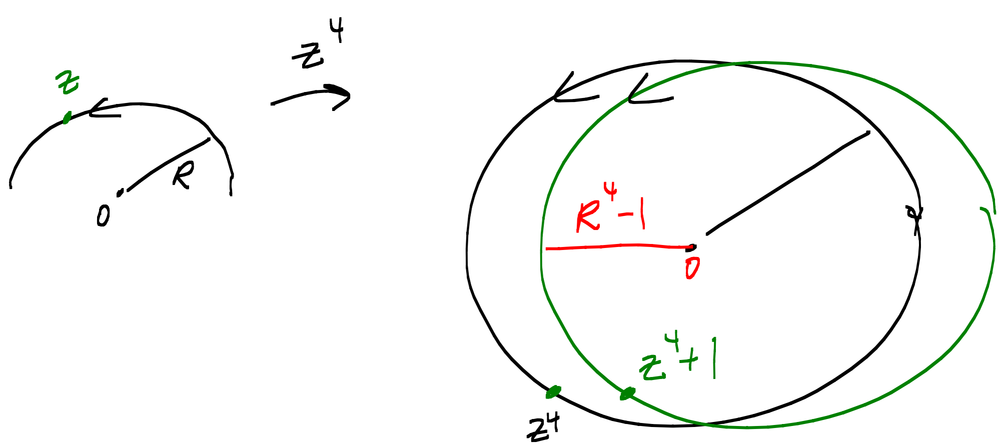
$$|a + B| \ge \big| |a| - |B| \big|$$

$$|z^4+1| = |z^4 - (-1)| \ge \Big| \underbrace{|z^4|}_{R^4} - \underbrace{|-1|}_{1} \Big| = R^4 - 1$$

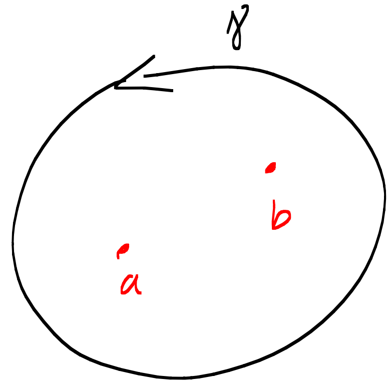
when $|z| = R$, $R > 1$

$$|I| \le \frac{\pi R}{R^4 - 1} \quad \text{when } R > 1$$

$$\to 0 \quad \text{as } R \to \infty.$$



EX:

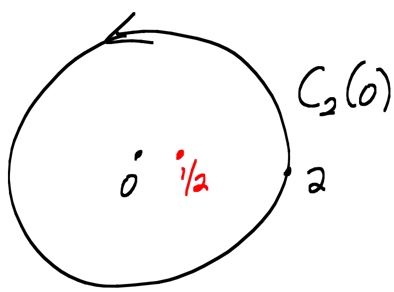


$$\frac{1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}$$

$$\int_\gamma \frac{f(z)}{(z-a)(z-b)}\,dz = A\int_\gamma \frac{f(z)}{z-a}\,dz + B\int_\gamma \frac{f(z)}{z-b}\,dz$$

Now use C.I. Formula.

EX:

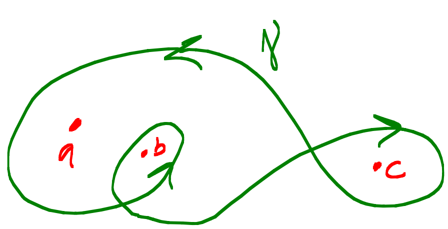


$$\int_{C_2(0)} \frac{e^{3z}}{\underset{\substack{\uparrow \\ \text{nice} \\ \text{inside } C_2(0)}}{(z-4)}\underbrace{(2z-1)^2}_{2^2(z-\frac{1}{2})^2}}\,dz$$

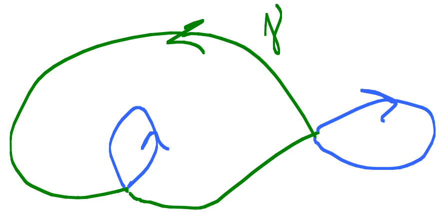
$$= \int_{C_2(0)} \frac{\left[\frac{e^{3z}}{4(z-4)}\right] \leftarrow f(z)}{(z-1/2)^2}\,dz = \frac{2\pi i}{1!} f'\left(\tfrac{1}{2}\right)$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

EX:



$$\int_\gamma \frac{e^{2z}}{z-w}\,dz \qquad w = a, b, c$$



$$\frac{1}{2\pi i}\int \frac{f(z)}{z-c}\,dz = -f(c)$$

clockwise