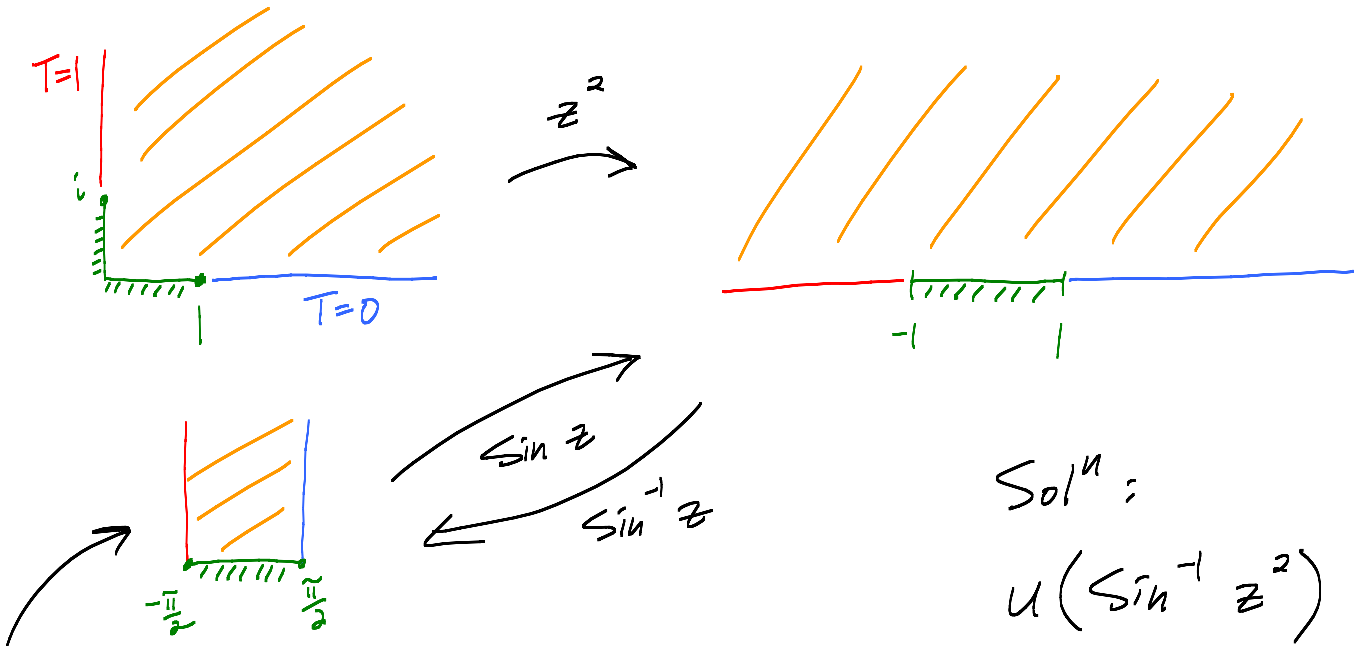
Review lecture 1



Sol^n:

$u(\sin^{-1} z^2)$

Sol^n here: $\underbrace{A \operatorname{Re} z + B}_{u(z)} = Ax + B$

Prob: Where does max $|z^4 + z^2|$ occur in $\overline{D_1(0)} = \{z : |z| \le 1\}$

Max princ: $z^4 + z^2$ not const. So max occurs on boundary

$|z| = 1$. $\quad |z^4 + z^2| = \underbrace{|z^2|}_{=|z|^2 = 1 \text{ on } C_1(0)} |z^2 + 1| = |(e^{i\theta})^2 + 1|$

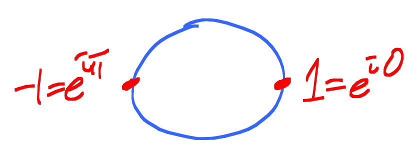
$$= |e^{i2\theta} + 1| = |(\cos 2\theta + i \sin 2\theta) + 1|$$

$$= \sqrt{(\cos 2\theta + 1)^2 + \sin^2 2\theta}$$

$$= \sqrt{2 + 2\cos 2\theta}$$

max = 2 at peaks of $\cos 2\theta$:

$2\theta = 2n\pi \qquad \theta = n\pi$



Suppose f and g are two entire functions such that g is not identically equal to zero and

$$|f(z)| \leq |g(z)|$$

for all z. Prove that $f = cg$ for some constant c with $|c| \leq 1$. (Be mindful of points where g might vanish.)

Hmmm. $\frac{f}{g}$ is analytic where g does not vanish.

Claim: zeroes of g are removable singularities for $\frac{f}{g}$.

Suppose z_0 is a zero of g.

Important: g not constant $\Longrightarrow$ z_0 is an isolated zero of g. So $\exists\, r > 0$ such that $g(z) \neq 0$ on $D_r(z_0) - \{z_0\}$.

Now $\frac{f}{g}$ analytic on $D_r(z_0) - \{z_0\}$ and $|f(z)| \leq |g(z)|$ yields that $\left|\frac{f}{g}\right| \leq 1$ there. So z_0 must be a removable sing for $\frac{f}{g}$.

Note: Value at z_0 has modulus ≤ 1 by contin.

Let $h(z) = \frac{f(z)}{g(z)}$ if $g(z) \neq 0$ and the value to make it analytic at zeroes of g.

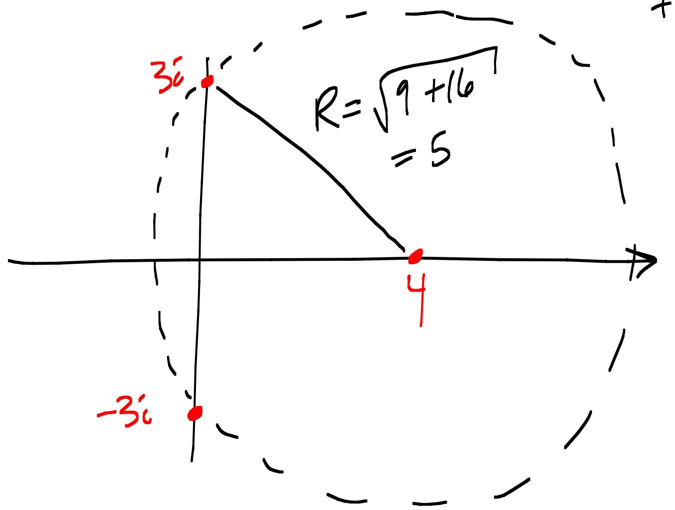
h is bdd entire. Liouville's $\Rightarrow$ $\frac{f}{g} \equiv c$.

where $g \neq 0$

$f \equiv cg$ $\Leftarrow$ where $g \neq 0$.

True where $g(z) = 0$ by continuity.

What is the radius of convergence of the power series for $\underbrace{\dfrac{\sin z}{z^2+9}}_{f(z)}$ centered at $z = 4$? Explain.



R of C ≥ 5.

And R of C ≤ 5 too because blows up at $\pm 3i$.

So $R=5$.

Harder prob: $f(z) = \dfrac{\operatorname{Log} z}{z-1}$ at $z=4$.

removable sing at $z=1$!

$R=4$

How many zeroes does the polynomial $z^4 + z^3 - z^2 + 2$ have inside the disc of radius two about the origin? Explain.

Arg princ: $\# \text{zeroes} = \dfrac{1}{2\pi i} \displaystyle\int_{C_2(0)} \frac{P'(z)}{P(z)}\, dz$

Better: Use Rouché's Thm. $|h(z)| < |f(z)|$ on $C_2(0)$ $\Longrightarrow$ f and $f+h$ have same # zeroes.

the big one $\rightarrow$ $f(z)$

$$\underbrace{z^4}_{f(z)} + \underbrace{z^3 - z^2 + 2}_{h(z)}$$

$|\ \ |$ on C_2: $\quad 2^4 = 16 \qquad 2^3 = 8 \qquad 2^2 = 4 \qquad 2$

$$|h(z)| = |z^3 - z^2 + 2| \le \underbrace{|z|^3 + |z|^2 + 2}_{2^3 + 2^2 + 2 = 14 \text{ on } C_2(0),}$$

$$< 16 = \underbrace{|z^4|}_{|f(z)|} \text{ on } C_2(0)$$

Rouché $\Longrightarrow$ $f+h$ has 4 zeroes inside $C_2(0)$.

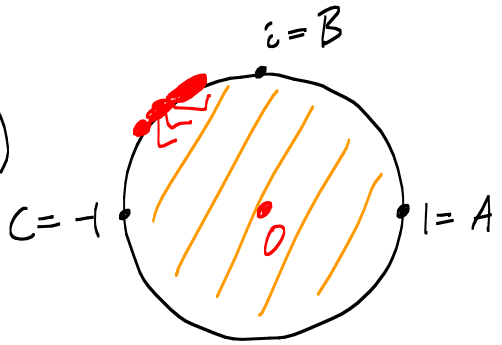
$|z^4 + z^3| \ge \big||z|^4 - |z|^3\big| = 16 - 8 = 8 \qquad h = -z^2 + 2$ works too.

What is the image of the unit disc under the following mappings?

a) $T_1(z) = z/(1-z)$ b) $T_2(z) = z/(z - i/2)$

$$= \frac{1 \cdot z + 0}{-z + 1}$$

a)

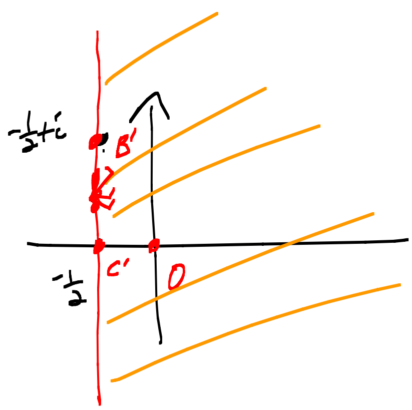


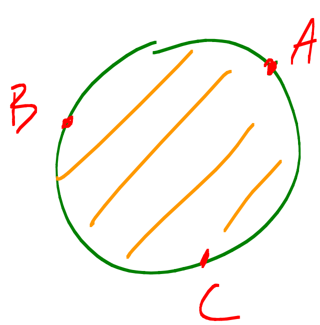
$T_1(1) = \infty \leftarrow A'$ ← so circle → line!

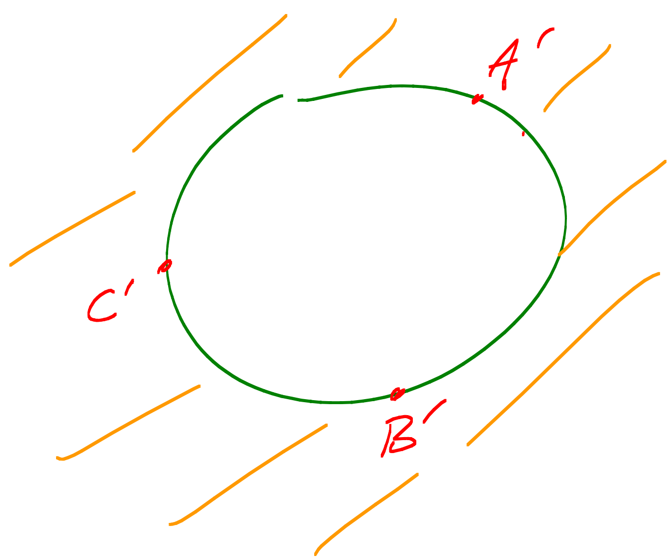
$T_1(-1) = \frac{-1}{1-(-1)} = \frac{-1}{2} \leftarrow C'$

$T_1(i) = \frac{i}{1-i} \cdot \frac{1+i}{1+i} = \frac{-1+i}{2}$

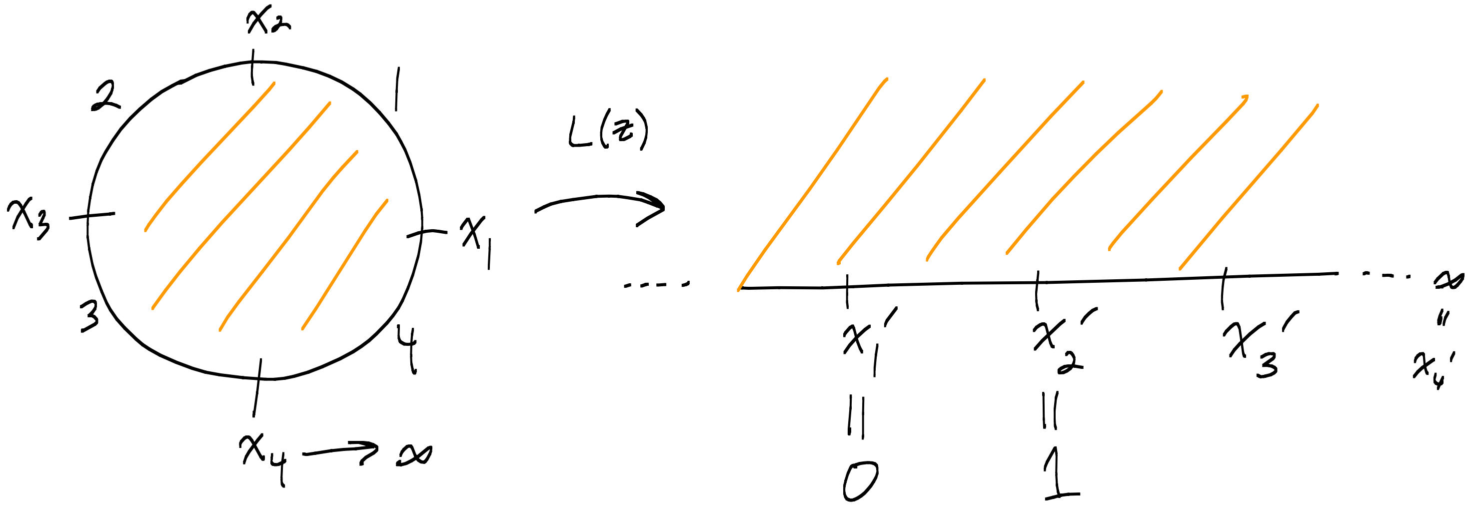
$T_1(0) = \frac{0}{1-0} = 0$







Find a harmonic function on the unit disc that has boundary values on the unit circle equal to one in the first quadrant, two in the second quadrant, three in the third quadrant, and four in the fourth quadrant.



$$\frac{i-(-i)}{i-1}\cdot\frac{z-1}{z-(-i)} = L(z)$$

$x_3' = L(-1)$.

Sol^n here:

$$A \operatorname{Arg} z + B \operatorname{Arg}(z-1) + C \operatorname{Arg}(z-x_3') + D$$