Review 2

Final Exam Friday, December 13, 1:00 - 03:00pm in WTHR 160

Let C_1 denote the unit circle parametrized in the counterclockwise sense. Evaluate the following integrals. Explain.

a) $\displaystyle\int_{C_1} \frac{e^{2z}}{z-4}\,dz = 0$ by Cauchy's Thm!

b) $\displaystyle\int_{C_1} \frac{e^{2z}}{4z^2+1}\,dz = 2\pi i\left(\operatorname{Res}_{\frac{i}{2}} \frac{e^{2z}}{4z^2+1} + \operatorname{Res}_{-\frac{i}{2}} \cdots\right)$

c) $\displaystyle\int_{C_1} \frac{e^{2z}}{2z^2-5z+2}\,dz = 2\pi i \cdot \frac{e^{2a_1}}{4a_1-5}$ a_1 zero inside

d) $\displaystyle\int_{C_1} \frac{e^{2z}}{(z-1/2)^5}\,dz = \int_{C_1} \frac{A_0}{(z-\frac{1}{2})^5} + \frac{A_1}{(z-\frac{1}{2})^4} + \cdots + \frac{A_4}{z-\frac{1}{2}} + \cdots\,dz$ $\quad A_4 = \left.\dfrac{\frac{d^4}{dz^4}e^{2z}}{4!}\right|_{z=\frac{1}{2}}$

e) $\displaystyle\int_{C_1} \frac{1}{z^5}\,dz = 0$ by Res. Thm or $\frac{1}{z^5} = \frac{d}{dz}\left(\frac{-1}{4}z^{-4}\right)$

$$\frac{e^{2z}}{4(z-\frac{i}{2})(z+\frac{i}{2})} = \frac{1}{z-\frac{i}{2}}\left[\frac{e^{2z}}{4(z+\frac{i}{2})}\right]$$

$\operatorname{Res}_a \dfrac{f}{g} = \dfrac{f(a)}{g'(a)}$ when a is simple zero of g

$\operatorname{Res}_a \dfrac{f(z)}{z-a} = f(a)$

f analytic near a.

Compute

$$\mathrm{Res}_i \frac{\mathrm{Log}\ z}{(z^2+1)} \qquad \text{and} \qquad \mathrm{Res}_i \underbrace{\frac{e^{2z}}{(z^2+1)^2}}.$$

$$\| $$

$$\frac{\mathrm{Log}\ i}{2i} = \frac{\mathrm{Ln}|i| + i\frac{\pi}{2}}{2i} = \frac{\pi}{4}$$

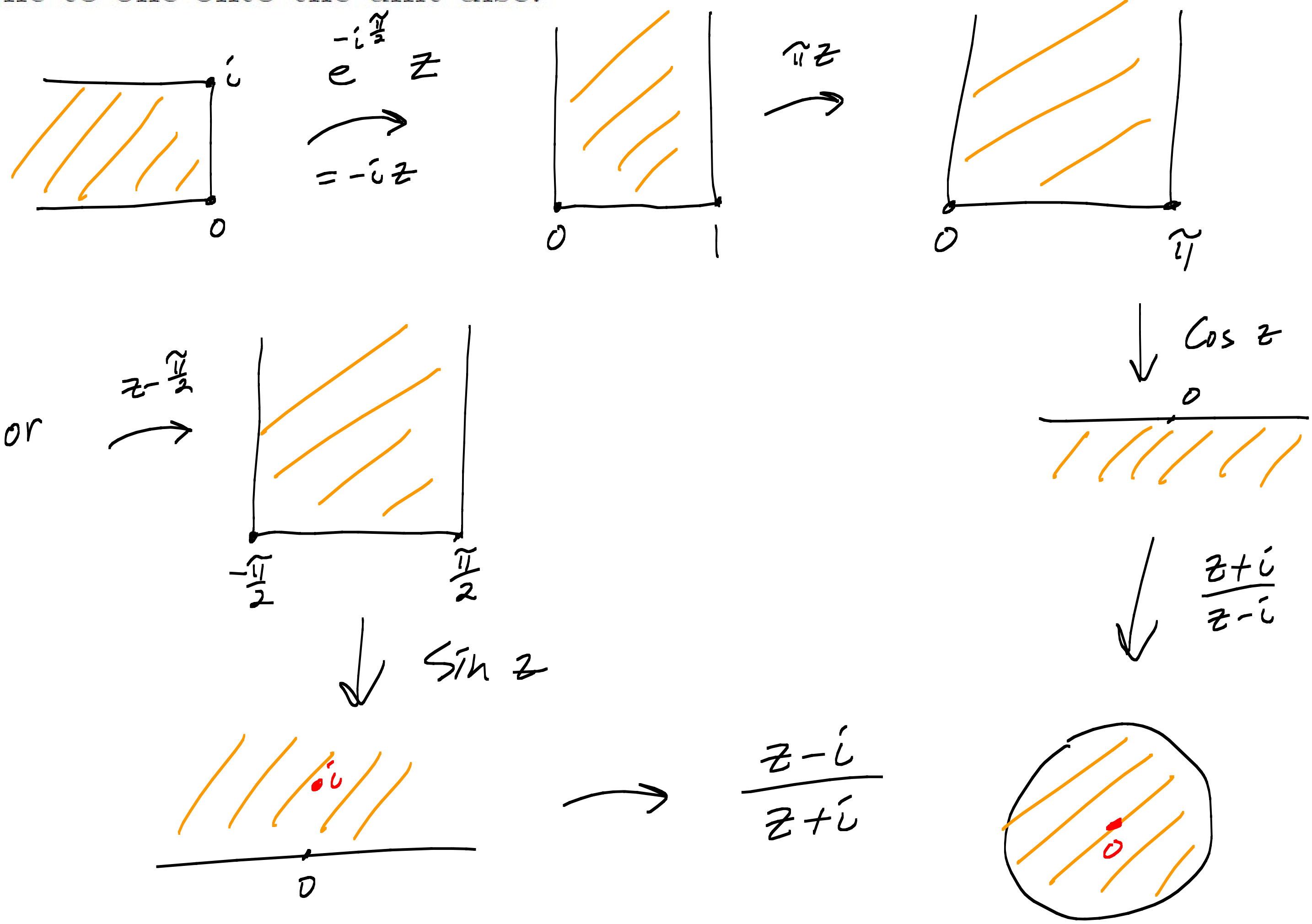
$$\frac{e^{2z}}{(z-i)^2(z+i)^2}$$

$$\frac{1}{(z-i)^2}\left[\underbrace{\frac{e^{2z}}{(z+i)^2}}\right]$$

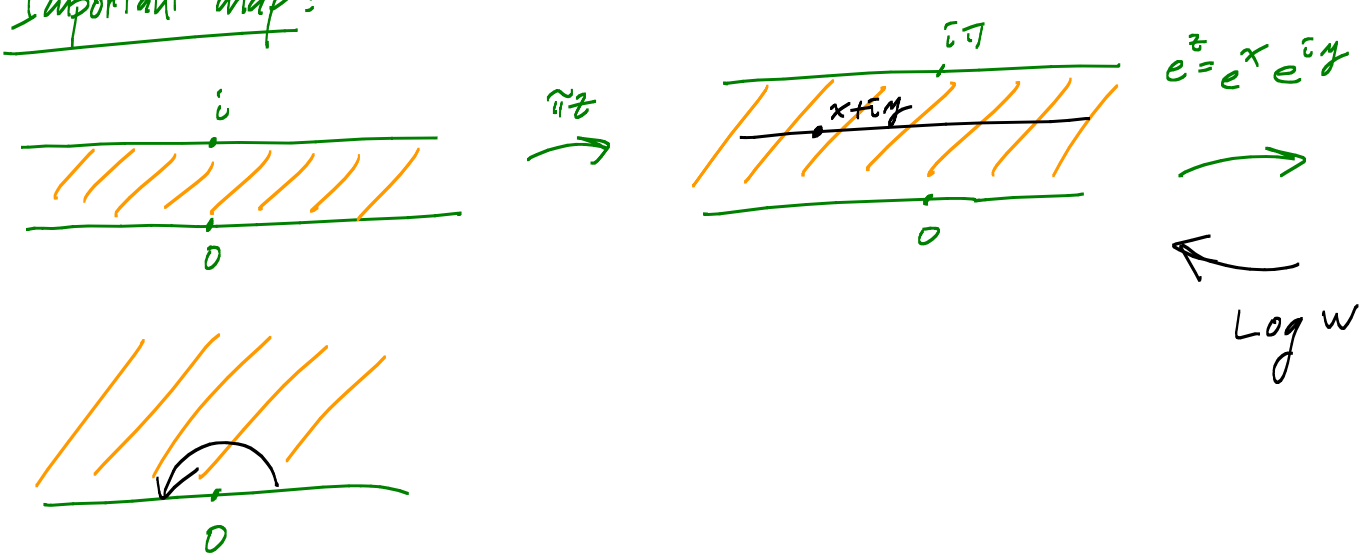
$$H(z) = A_0 + A_1(z-i) + \cdots$$

$$= \frac{A_0}{(z-i)^2} + \frac{A_1}{z-i} + \cdots \qquad A_1 = \frac{H'(i)}{1!}$$

Find an analytic functions that maps the semi-infinite rectangle
$\{x + iy : -\infty < x < 0, \quad 0 < y < 1\}$
one-to-one onto the unit disc.

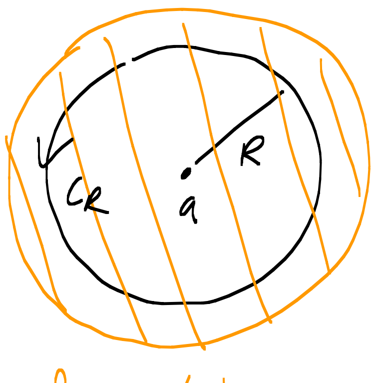


Important map:



State Liouville's Theorem and show how the Cauchy estimate for the first derivative implies it. (If you don't remember the Cauchy estimate for the first derivative, you can deduce it by differentiating the Cauchy integral formula for a disc and estimating the derivative at the center using the basic estimate.)

L: Bdd entire fcns must be const.



$$\left|f^{(n)}(a)\right| \leq \frac{n!\,M}{R^n}$$

$$M = \max_{|z-a|=R} |f|$$

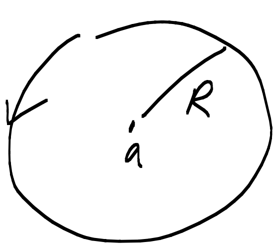
$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C_R(a)} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Basic estimate: $\left|f^{(n)}(a)\right| \leq \frac{n!}{2\pi}\left(\max_{|z-a|=R} \frac{|f(z)|}{R^{n+1}}\right) \underbrace{\text{Length}(C_R)}_{2\pi R}$

✓ $\frac{n!\,M}{R^n}$

First order est: f bdd entire: $|f(z)| \leq M$ on $\mathbb{C}$.

Pick $a \in \mathbb{C}$.



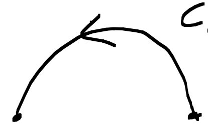
$$|f'(a)| \leq \frac{1!\,M}{R^1}$$

let $R \nearrow \infty$.

See $|f'(a)| = 0$.

a arbitrary. So $f' \equiv 0$. $\Rightarrow$ f const.

Careful estimating $\int$'s.



C_R

$$\left| \int_{C_R} \frac{1}{z^5+1}\, dz \right| \le \frac{1}{R^5-1} \cdot \pi R \quad \text{if} \quad R > 1$$

Denom est: $|z^5+1| \ge \underbrace{\big|\, |z^5| - |-1| \,\big|}$

$$= |R^5 - 1| = R^5 - 1 \quad \text{if} \quad R > 1.$$

Note: $|I| \le \dfrac{1}{1-\varepsilon^5} \cdot \pi\varepsilon$ if $0 < \varepsilon < 1$ $(R = \varepsilon)$

Or Use the Basic Polynomial Estimate:

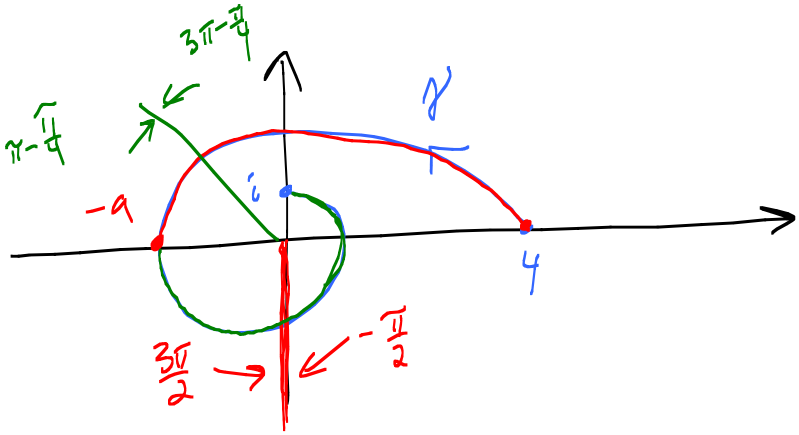
$P(z)$ degree $N \ge 1$.

$$\underbrace{a|z|^N \le |P(z)|}_{\text{denom est}} \le A|z|^N \quad \text{when} \quad |z| > R_0$$

Then $\left| \dfrac{1}{z^5+1} \right| \le \dfrac{1}{a|z|^5} = \dfrac{1}{aR^5}$ if $\underline{\underline{R > R_0}}$

$R = |z|$

Way wrong: $\left| \dfrac{1}{z^5+1} \right| \le \dfrac{1}{R^5+1}$



$$\int_\gamma \frac{1}{z}\,dz = \left[\log(-a) - \log 4\right] + \left[\log i - \log(-a)\right]$$

$$= \log i - \log 4$$

$$= \left[\ln|i| + i\left(2\pi + \frac{\pi}{2}\right)\right] - \left[\ln 4 + i\cdot 0\right]$$

$$= \left(\ln|i| - \ln 4\right) + i\underbrace{\left(2\pi + \frac{\pi}{2}\right)}_{\text{Total } \Delta \text{Arg}}$$