Review for Exam 2 (in class Wed 11/13)

8. Suppose that f and g are analytic on a disc about a point a and suppose that g has a zero of multiplicty two at a. Show that the residue of f/g at a is equal to

$$\frac{6g''(a)f'(a) - 2g'''(a)f(a)}{3g''(a)^2}.$$

Simple zero

$$\operatorname{Res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$$

$$g(z_0) = 0$$
$$g'(z_0) = 0$$
$$g''(z_0) \neq 0$$

$$g(z) = a_2(z-z_0)^2 + \cdots$$

$$= (z-z_0)^2 \underbrace{\left[a_2 + a_3(z-z_0) + \cdots\right]}_{G(z)}$$

$$* \left[\begin{array}{l} G(z_0) = a_2 = \dfrac{g''(z_0)}{2!} \\[2ex] \dfrac{G'(z_0)}{1!} = a_3 = \dfrac{g'''(z_0)}{3!} \end{array}\right.$$

$$\frac{f(z)}{g(z)} = \frac{1}{(z-z_0)^2}\left[\underbrace{\frac{f(z)}{G(z)}}_{A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \cdots}\right]$$

$$= \frac{A_0}{(z-z_0)^2} + \frac{A_1}{z-z_0} + A_2 + \text{p.s.} \cdots$$

Res

$$\operatorname{Res}_{z_0} \frac{f}{g} = A_1 = \frac{\frac{d}{dz}\left[\frac{f}{G}\right]\Big|_{z=z_0}}{1!} = \frac{f'G - G'f}{G^2}\Big|_{z=z_0}$$

Now use $(*)$

1. Let C denote the unit circle parameterized in the counterclockwise direction. Compute

$$\int_C \frac{e^{2z}}{2z^2+5z+2}\,dz. \quad = 2\pi i \,\mathrm{Res}_{r_1} H(z) = 2\pi i \,\frac{e^{2r_1}}{4r_1+5}$$

Roots $2z^2+5z+2$ are r_1, r_2. ← r_1 inside C, r_2 outside

$$H = \frac{f}{g} \leftarrow \text{simple zero at } r_1$$

$g(r_1) = 0$ $\qquad r_1 = -\frac{1}{2}$

$g'(z) = 4z + 5$ $\qquad r_2 = -2$

Check: $g'(r_1) \neq 0$.

$$\frac{e^{z}}{(2z+1)(z+2)} = \frac{e^{z}}{2\left(z-\left(-\frac{1}{2}\right)\right)(z+2)} = \frac{1}{z-\left(-\frac{1}{2}\right)}\left[\underbrace{\frac{e^{z}}{2(z+2)}}_{G(z)}\right]$$

$$\mathrm{Res}_{-\frac{1}{2}} = G\left(-\tfrac{1}{2}\right)$$

$$\mathrm{Res}_a \frac{f(z)}{z-a} = f(a)$$

3. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{5n+7} z^{3n}.$$

u_n

Book: $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ ← true if limit exists! Let $w = z^3$

Better: $\left|\frac{u_{n+1}}{u_n}\right| = \left| \frac{\frac{2^{n+1}}{5(n+1)+7} z^{3(n+1)}}{\frac{2^n}{5n+7} z^{3n}} \right| = 2 \cdot \frac{5n+7}{5n+12} \cdot |z^3|$

$\xrightarrow[n\to\infty]{} 2|z|^3$ < 1 Conv. > 1 Div.

$|z| < \frac{1}{2^{1/3}}$ Conv. Div outside. $R = \frac{1}{2^{1/3}}$

What if I replace z^{3n} by z^{n^2}? Must use Ratio test!

$\left|\frac{u_{n+1}}{u_n}\right| = 2 \frac{5n+7}{5n+12} \underbrace{\left|\frac{z^{(n+1)^2}}{z^{n^2}}\right|}_{|z|^{2n+1}}$ $\to 0$ if $|z| < 1$, $\to \infty$ if $|z| > 1$

Conv when $|z| < 1$. Div $|z| > 1$. $R = 1$.

4. Calculate the residue at $z = 2i$ for the function

$$\frac{\text{Log } z}{(z^2+4)^2}, = \frac{\text{Log } z}{[(z-2i)(z+2i)]^2}$$

where Log z denotes the principal branch of the complex log function. Express your answer as a complex number $a + bi$ in simplest terms (i.e., evaluate things like Log $2i$ in your answer and simplify).

$$= \frac{1}{(z-2i)^2}\left[\frac{\text{Log } z}{(z+2i)^2}\right]$$

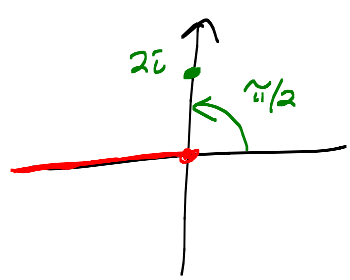
$$A_0 + A_1(z-2i) + \cdots$$

$$= \frac{A_0}{(z-2i)^2} + \frac{A_1}{z-2i} + \text{p.s.} + \cdots$$

$$\text{Res}_{2i} = \frac{\left.\frac{d}{dz}\left[\frac{\text{Log } z}{(z+2i)^2}\right]\right|_{z=2i}}{1!}$$

<u>Note</u>: $\frac{d}{dz}\text{Log } z = \frac{1}{z}$ ← same for any branch

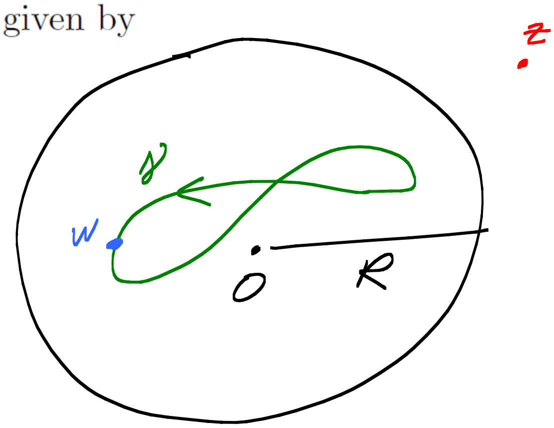
Log is <u>principal</u> branch



$$\text{Log } 2i = \text{Ln}|2i| + i\,\text{Arg}\, 2i = \text{Ln}\, 2 + i\frac{\pi}{2}$$

5. Let γ denote a closed curve inside the disc of radius 5 about the origin and suppose that ϕ is a continuous complex valued function on γ. Let $F(z)$ denote the complex function defined for z on $\mathbb{C}$ minus the trace of γ given by

$$F(z) = \int_\gamma \frac{\phi(w)}{w - z}\, dw.$$



Use careful estimates to show that

$$\lim_{z \to \infty} F(z) = 0.$$

Basic estimate: $$|F(z)| \le \left(\max_{w \in tr(\gamma)} \frac{|\varphi(w)|}{|w-z|} \right) \text{Length}(\gamma)$$

$|\varphi(z(t))|$ is continuous on $[a,b]$. So it attains its max M.

Denom estimate: $|w-z| \ge \big||w| - |z|\big| = |z| - |w|$ if $|z| > R$,

$\ge |z| - R$ if $\begin{array}{l}|z| > R \\ |w| < R\end{array}$

So $$|F(z)| \le \frac{M \text{ Length}(\gamma)}{|z| - R} \qquad |z| > R. \quad (R=5).$$

$\longrightarrow 0$ as $|z| \to \infty$.

F has a removable sing at ∞. In fact, it has a zero at ∞.

1. Prove that if a twice continuously differentiable harmonic function vanishes on an open subset of a domain, then it must be zero on the whole domain. Hint: If u is harmonic, then $\underbrace{u_x - iu_y}_{f(z)}$ is analytic.

C-R Eqn #1 : $u_{xx} = -u_{yy}$ ✓

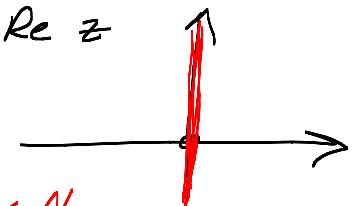
#2 : $u_{xy} = u_{yx}$ ✓

Hyp $\Rightarrow$ f vanishes on $D_r(z_0) \subset \Omega$. Identity Thm $\Rightarrow$

$f \equiv 0$ on Ω. $\Rightarrow$ $u_x \equiv 0$, $u_y \equiv 0$ $[\nabla u \equiv 0]$.

$\Rightarrow$ $u \equiv$ const.

$x = \operatorname{Re} z$



Careful: No iden. thm. for harm. fcns.

2. Suppose that f and g are analytic on a disc about a and that each has a zero of multiplicity N at a. Prove a L'Hôpital's formula: $\lim_{z\to a} \dfrac{f(z)}{g(z)} = \dfrac{f^{(N)}(a)}{g^{(N)}(a)}$.

$$f(z) = a_N (z-a)^N + \cdots = (z-a)^N \underbrace{[a_N + a_{N+1}(z-a) + \cdots]}_{F(z).}$$

$F(a) = a_N = \dfrac{f^{(N)}(a)}{N!}$

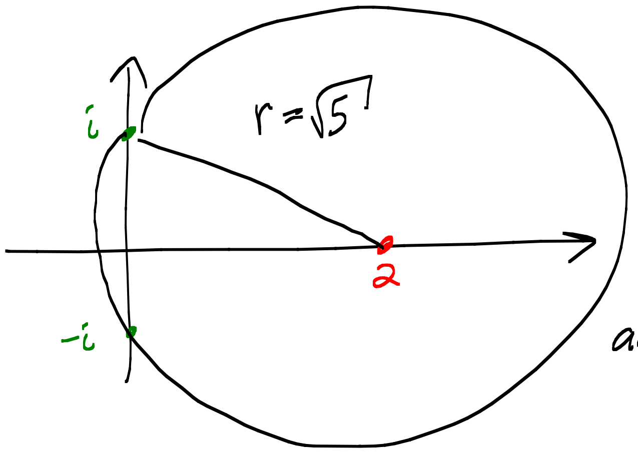
$$g(z) = b_N (z-a)^N + \cdots = (z-a)^N \underbrace{[b_N + \cdots]}_{G(z)}$$

$G(a) = \dfrac{g^{(N)}(a)}{N!}$

$$\frac{f}{g} = \frac{F}{G} \xrightarrow[z\to a]{} \frac{F(a)}{G(a)} = \frac{f^{(N)}(a)}{g^{(N)}(a)}$$ ✓

3. Let $f(z) = \frac{1}{1+z^2}$. Find the first three terms in the power series $\sum_{n=0}^{\infty} a_n (z-2)^n$ for f centered at $z = 2$. What is the radius of convergence of the series? Explain.

$a_n = \frac{f^{(n)}(2)}{n!}$ $\quad n = 0, 1, 2 \quad$ not hard.



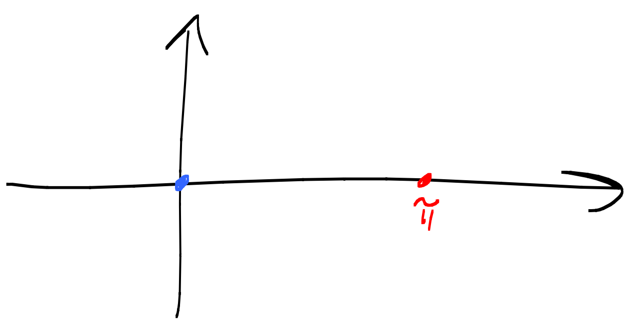
So $\underline{R \text{ of } C \geq \sqrt{5}}$.

analytic on disc

But $f(z)$ blows up at poles $\pm i$. So $R \leq \sqrt{5}$ too.

$R = \sqrt{5}$.

4. What is the radius of convergence of the power series for $\frac{\sin z}{z}$ centered at $z = \pi$?



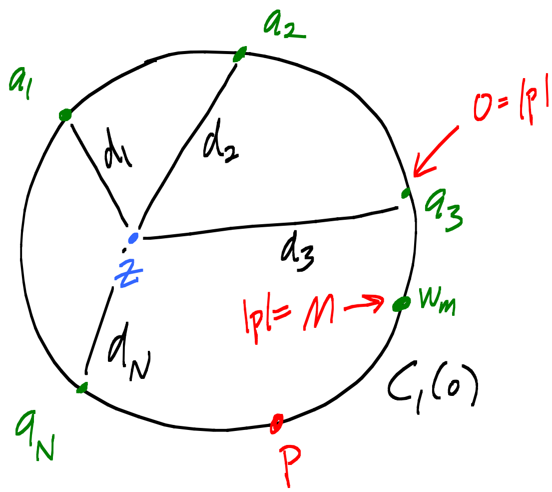
Aha! $\frac{\sin z}{z}$ has a <u>removable</u> sing at $z = 0$.

$$h(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

is entire.

R of C for power series for h about π is ∞.

6. Suppose $a_1, a_2, \ldots, a_n$ are points on the unit circle. For a point z on the unit circle, let $D(z)$ denote the product of the distances from z to each of the n points on the circle. Prove that there is a point z on the circle such that the product $D(z)$ is exactly equal to one. Hint: Let $p(z)$ denote the product of $(z - a_n)$ as n ranges from one to n. What is $|p(0)|$? Use the Maximum Modulus Principle.



$$D = \prod_{k=1}^{N} d_k$$

$D(p) = 1$ exists.

$$p(z) = \prod_{k=1}^{N} (z - a_k)$$

$|p(z)| = D(z)$ $\quad$ p poly of deg $N > 1$. p is not const.

$|p(0)| = \prod \underbrace{|a_k|}_{1} = 1$ $\leftarrow$ can't be max modulus (otherwise $p \equiv$ const. ↯)

Max Modulus Thm $\Rightarrow$ $\exists$ $w_m \in C_1(0)$ where $|p(w_m)| = M > 1$

max modulus.

$|p(a_k)| = 0$, $|p(w_m)| = M > 1$

Intermediate value thm.

$\Rightarrow$ $\exists$ place between a_k and w_m on $C_1(0)$

where $|p| = 1$. ✓

9. Assume $b_0 = 1$. Constants $b_2, b_4, b_6, \ldots$ are generated via the recursion formula

$$b_{n+2} = \frac{4(n+3)(n-5)}{7(n+1)(n+2)}\, b_n.$$

What is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} b_{2n} z^{2n} = b_0 + b_2 z^2 + b_4 z^4 + \cdots ?$$

$$\lim_{\text{out} \to \text{way out}} \left| \frac{u_{\text{out there}}}{u_{\text{before}}} \right|$$

and use <u>recursion</u> formula

10. If the complex numbers a_n tend to zero as n tends to infinity, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is at least one.

Hadamard's: $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{N\to\infty} \sup_{n\geq N} \sqrt[n]{|a_n|}$

$0 \leq |a_n| < 1$ if $n > N_0$ since $a_n \to 0$.

Now take n-th root: $0 \leq \sqrt[n]{|a_n|} < 1$. Aha! Lim of Sups has to be ≤ 1.

So $\frac{1}{R} \leq 1$.

And $R \geq 1$. ✓

Or better yet, since $a_n \to 0$, it is a bounded seq.

So $\exists\ M$ such that $|a_n| \leq M$ for all n.

Now, if $|z| < 1$, then $|a_n z^n| = |a_n||z|^n \leq \underbrace{M|z|^n}_{\text{Conv. Geom Series}}$

Comparison test shows $\sum a_n z^n$ conv if $|z| < 1$. So $R \geq 1$.