MA 425 Complex analysis

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$x^2+1=0$. Toss in i such that $i^2=-1$.

Also need $3+i$, $3+7i$, $\pi+ei$, $(3+4i)^{10} = \cdots = a+bi$

i, $i^2=-1$, $i^3=-i$, $i^4=1$, ...

$\mathbb{C} = \{a+bi : a,b \in \mathbb{R}\} \approx \{(x,y) : x,y \in \mathbb{R}\}$

$$z_1+z_2 = (x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2)$$

$$(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$$

$$z_1\cdot z_2 = (x_1x_2-y_1y_2)+i(x_2y_1+x_1y_2)$$

$$(x_1,y_2)\cdot(x_1,y_2) = (x_1x_2-y_1y_2, x_2y_1+x_1y_2)$$

Zero element: $0+0i$ ⟵ write 0 $\qquad (0,0)$

Identity element: $1+0i$ ⟵ write 1 $\qquad (1,0)$

i such $i^2=-1$ $\qquad (0,1)$

Big fact: $\mathbb{C}$ is a field.

Multiplicative inverse: $z^{-1} = \dfrac{1}{z} = \dfrac{1}{x+iy}\,\dfrac{x-iy}{x-iy}$

$$= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

Fun exercise: $z\cdot w = 1$

$(x+iy)(u+iv)=1$

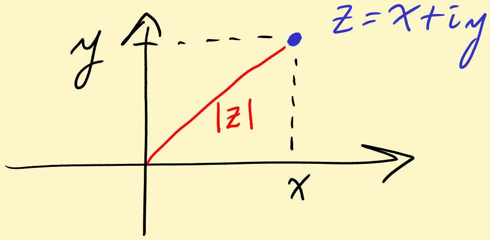
$$\underbrace{(ux - vy)}_{1} + i\,\underbrace{(vx + uy)}_{0} = 1$$

Cramer's rule : $x = \dfrac{\det\begin{pmatrix}1 & -v\\ 0 & u\end{pmatrix}}{\det\begin{pmatrix}u & -v\\ v & u\end{pmatrix}} = \dfrac{u}{u^2+v^2}$

$$y = \cdots$$

See inverse is unique.

Notation



$x = \operatorname{Re} z$

$y = \operatorname{Im} z$

$|z| = \sqrt{x^2 + y^2}$

← the modulus of z

$\bar{z} = x - iy$

← the complex conjugate of z

Facts

$\overline{(zw)} = \bar{z}\,\bar{w}$

$\overline{(1/z)} = 1/\bar{z}$

$z\bar{z} = |z|^2 \leftarrow \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$

Quadratic eqns : $x^2 + bx + c = 0$

$$\left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) = 0$$

$$\left(x + \frac{b}{2}\right)^2 = \underbrace{\frac{b^2}{4} - c}$$

ouch, if < 0.

Toss in i with $i^2 = -1$

Fundamental theorem of Algebra. Given any complex polynomial

$$P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$$

$a_N \neq 0$ $\quad a_n \in \mathbb{C}$

of degree $N \geq 1$, $P(z)$ has a zero in $\mathbb{C}$.

[zero of $P(z)$ = $r \in \mathbb{C}$ with $P(r) = 0$.]

Cor $P(z)$ factors over $\mathbb{C}$:

$$P(z) = a_N (z - r_1)(z - r_2) \cdots (z - r_N)$$

$r_n \in \mathbb{C}$. $\leftarrow$ r_n's can repeat

Exciting thing: Calculus with complex #s.

$$DQ = \frac{f(z) - f(a)}{z - a} \xrightarrow[z \to a]{} L, \text{ a limit.}$$

Say $f'(a) = L$. $\leftarrow$ complex derivative

Ex: $f(z) = z^2$. $\qquad DQ = \dfrac{z^2 - a^2}{z - a} = \dfrac{(z-a)(z+a)}{z-a} = z + a$

$\longrightarrow a + a = 2a$

as $z \to a$.

So $f'(a) = 2a$.

EX: $f(z) = \bar{z} = x - iy$ ← not $\mathbb{C}$-diff'ble!

Complex exponential $\quad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$$= e^{x+iy} = e^x \underbrace{e^{iy}}_{1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots}$$

$$= e^x \left[\underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right)}_{\cos y} + i \underbrace{\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}_{\sin y} \right]$$

Def^n: $e^{x+iy} = \left(e^x \cos y\right) + i\left(e^x \sin y\right)$

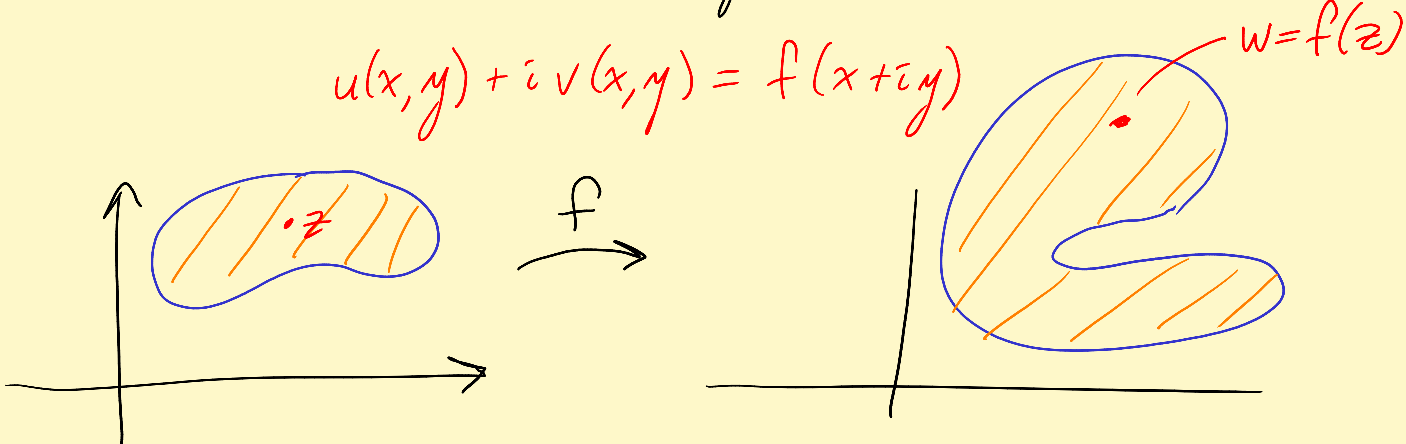
Most important fcn in all of math!

Feeling about $\quad w = f(z)$ $\qquad z = x + iy$

$w = u + iv$

$u + iv = f(x + iy)$

$u(x,y) + i\,v(x,y) = f(x+iy)$



Complex Diff'ble, $f'(z) \neq 0$, $\Rightarrow$ Conformal!