Lesson 2 Basic facts, complex exponential

Def^n: $e^z = e^{x+iy} = (e^x \cos y) + i(e^x \sin y)$

Properties

1) $e^0 = 1$
2) $\frac{d}{dz}(e^z) = e^z$

determine e^z

3) $e^{z+w} = e^z e^w$
4) $e^{-z} = 1/e^z$ ← so e^z is never zero

DeMoivre's Formula: $(e^{i\theta})^N = e^{iN\theta}$

Why $(e^{i\theta})^N = \underbrace{e^{i\theta} \cdot e^{i\theta} \cdots e^{i\theta}}_{N\text{-times}} = e^{i\theta + i\theta + \cdots + i\theta} = e^{iN\theta}$

Check today: $\underbrace{e^{i(\alpha+\beta)}}_{\cos(\alpha+\beta) + i\sin(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$

$$\cos(\alpha+\beta) + i \sin(\alpha+\beta)$$

$$= [\cos\alpha \cos\beta - \sin\alpha \sin\beta] + i[\cos\alpha \sin\beta + \sin\alpha \cos\beta]$$

$$= (\cos\alpha + i \sin\alpha)(\cos\beta + i \sin\beta) = e^{i\alpha} e^{i\beta} \checkmark$$

Pf of (3): $e^z e^w = e^{x+iy} e^{u+iv}$

$$= e^x e^{iy} e^u e^{iv} = (e^x e^u)(e^{iy} e^{iv})$$

$$= e^{(x+u)} e^{i(y+v)} = e^{z+w} \checkmark$$

Cool thing: Get trig identities via algebra!

Ex: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\sin 2\theta = \operatorname{Im} e^{i2\theta} = \operatorname{Im}\left[e^{i\theta}\cdot e^{i\theta}\right]$$

$$\cdots = \operatorname{Im}\left[(\text{eeee}) + i\,(2\sin\theta\cos\theta)\right]$$

$$= 2\sin\theta\cos\theta \checkmark$$

Arithmetic on $\mathbb{C}$: Algebra – just like on $\mathbb{R}$.

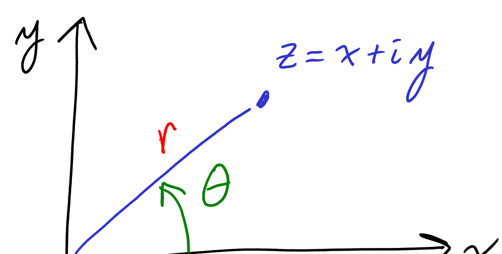
$\mathbb{C}$ is a field,

$\mathbb{C}$ is "complete", meaning every Cauchy seq in $\mathbb{C}$ has a limit.

$\mathbb{C}$ is "algebraically closed" meaning every poly over $\mathbb{C}$ has a root in $\mathbb{C}$ (unlike $\mathbb{R}$).

Analysis on $\mathbb{C}$: Topology just like $\mathbb{R}^2$.

Complex #'s in polar form via e^z



$x = \operatorname{Re} z = r\cos\theta$

$y = \operatorname{Im} z = r\sin\theta$

$r = |z| = \sqrt{x^2+y^2}$

$$z = x + iy = (r\cos\theta) + i\,(r\sin\theta) = \boxed{re^{i\theta}}$$

polar form of z

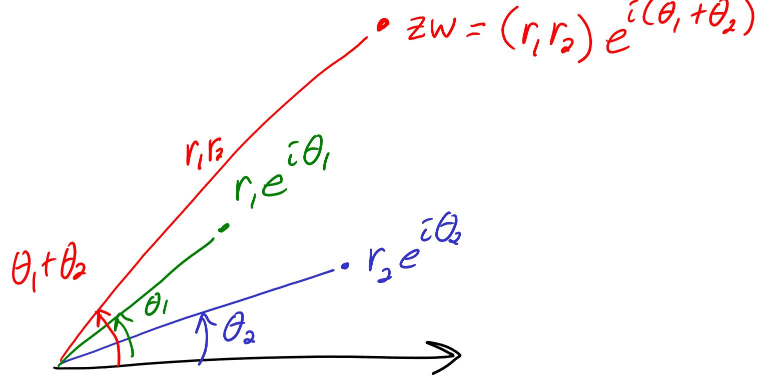
$\theta = \underline{\underline{A}}\text{rg } z$ = Principal argument

$= \theta$ with $-\pi < \theta \leq \pi$

$\arg z = \{\theta : z = |z| e^{i\theta}\} = \{\theta = \text{Arg } z + n 2\pi,\ n \in \mathbb{Z}\}$

Adding z and w in $\mathbb{C}$ is just vector addition in $\mathbb{R}^2$.

Complex multiplication



<u>Facts</u> $\quad |e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

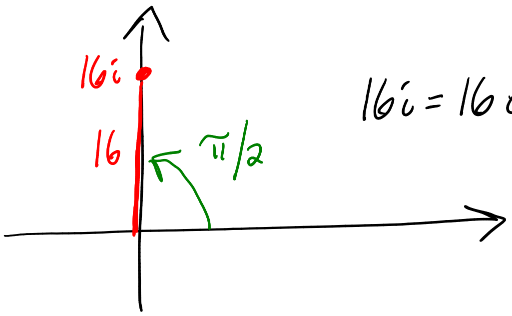
$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta)$$
$$= \cos\theta - i \sin\theta = \overline{e^{i\theta}}$$

$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$

<u>Finding complex roots</u>

<u>EX</u> Find all z such that $z^4 = 16i$

$$(\underbrace{re^{i\theta}}_{z})^4 \overset{\text{want}}{=} 16i$$



$16i = 16e^{i\pi/2}$

$$r^4 e^{i4\theta} = 16 e^{i\pi/2}$$

See: Need $r^4 = 16$

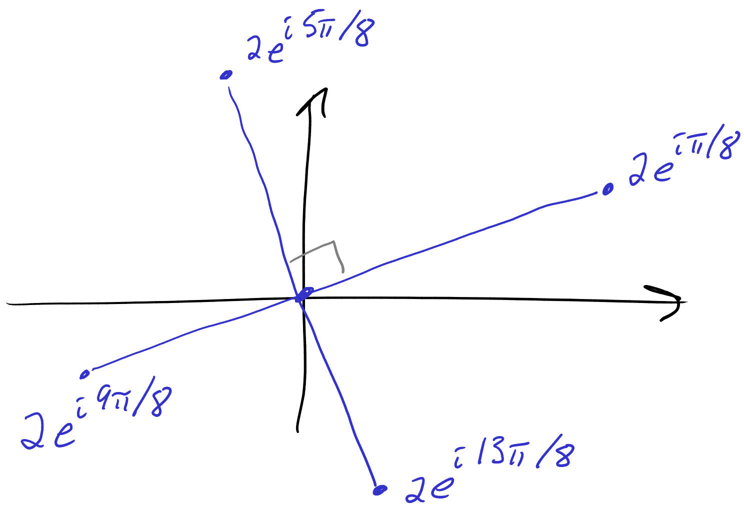
$r > 0$. So $r = 2$.

Hmmm. arguments 4θ and $\frac{\pi}{2}$ must represent same spot in $\mathbb{C}$.

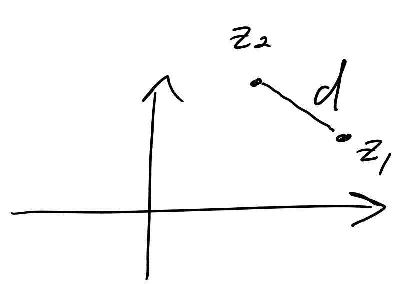
$$4\theta = \frac{\pi}{2} + n2\pi \text{ for some } n \in \mathbb{Z}$$

$$\boxed{\theta = \frac{\pi}{8} + \frac{n\pi}{2}} \quad n \in \mathbb{Z}$$

Ans: $z = re^{i\theta} = 2e^{i\left(\frac{\pi}{8} + n\frac{\pi}{2}\right)}$ ← only 4 pts!

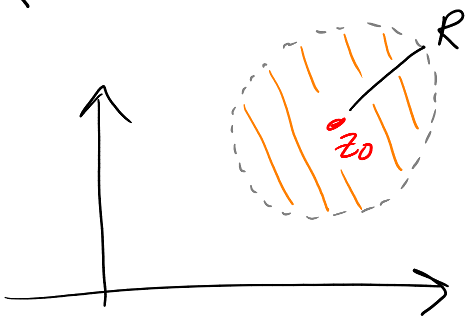


Topology on $\mathbb{C} \approx \mathbb{R}^2$



$$d = dist(z_1, z_2) = |z_1 - z_2|$$

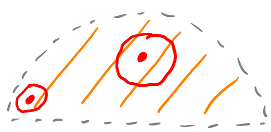
$D_R(z_0) = \{z : |z - z_0| < R\}$ = open disc of radius R about z_0



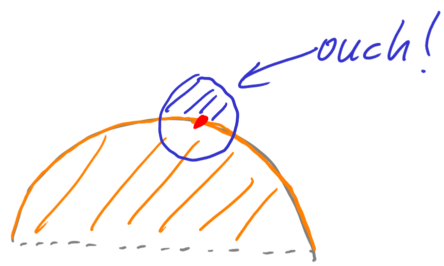
<u>Def^n</u> $\Omega \subset \mathbb{C}$ is called <u>open</u> if, for any $z_0 \in \Omega$, there is a disc $D_R(z_0) \subset \Omega$ (with $R > 0$).

Note: R can depend on z_0.

EX



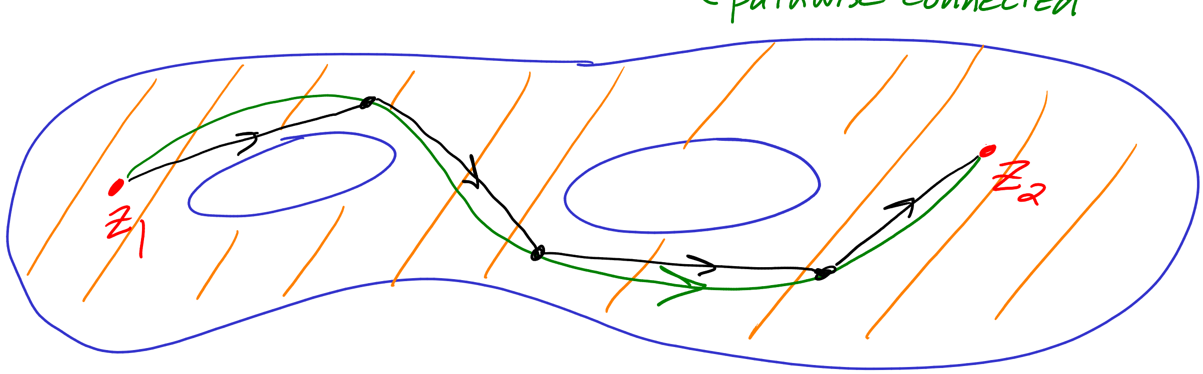
open



not open

Important Ω's : <u>Connected</u> open sets.

pathwise connected



Ω = orange shading, no blue bndry.

Any 2 pts in Ω can be connected by a $\underbrace{\text{curve}}_{\text{piecewise } C^1}$ <u>in</u> Ω.