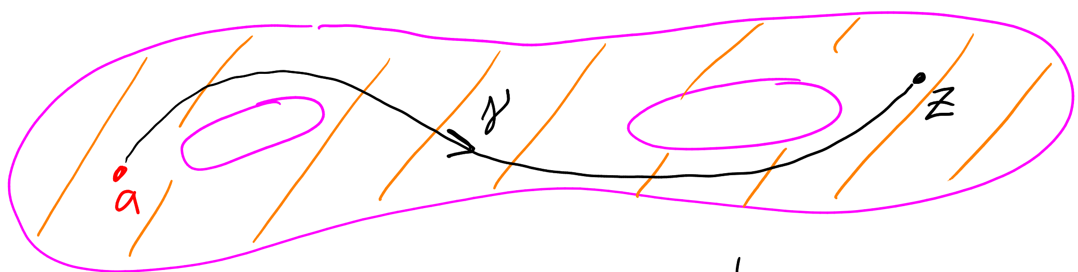
Lesson 3 Topology of $\mathbb{C}$, calculus on $\mathbb{C}$.

Read Chap 1. HWK 1 due Wed in Gradescope by 11:30pm

Def^n: An open connected set in $\mathbb{C}$ is called a domain.

Lemma Suppose $u(x,y)$ is continuously diff'ble on a domain Ω, $\nabla u \equiv 0$. Then u is constant on Ω.



γ piecewise $\mathcal{C}^1$

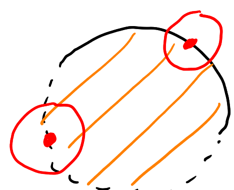
$$0 = \int_\gamma \underbrace{\nabla u}_{\equiv 0} \cdot d\vec{s} = u(z) - u(a).$$

$$u(z) = u(a)$$

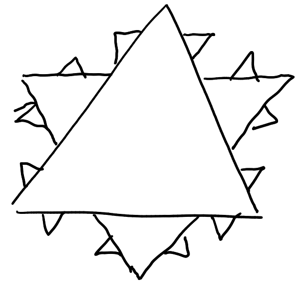
z: slide z around; a: fixed

Fact. If Ω_1, Ω_2 open, so are $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2$.

Def^n A point z_0 is a boundary pt of a set $S \subset \mathbb{C}$ if, for every $\varepsilon > 0$ (no matter how small), $D_\varepsilon(z_0)$ contains a pt in S and a pt outside S.



Sierpinski snowflake



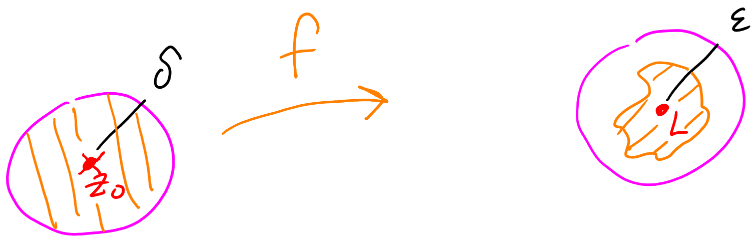
middle thirds are bases.

Calculus on $\mathbb{C}$

Limits $\lim_{z \to z_0} f(z) = L$ means:

Given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\underbrace{|f(z) - L| < \varepsilon}_{f(z) \in D_\varepsilon(L)} \quad \text{when} \quad \underbrace{|z - z_0| < \delta,\ z \neq z_0}_{z \in D_\delta(z_0) - \{z_0\}}.$$



Def^n f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Note: f is continuous on an open set if it is cont at each pt in the set.

Fact: All rules about limits and cont. fcns $\mathbb{R} \to \mathbb{R}$ carry over $\mathbb{C} \to \mathbb{C}$ with same proofs!

EX: If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then $\lim_{z \to z_0} f(z)g(z)$ exists and $= AB$.

And $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ provided $B \neq 0$.

Consequence: Complex polys are cont.

Why: $\underbrace{\underset{\text{cont}}{a_0} + \underset{\text{cont}}{a_1}\, \underset{\text{cont}}{z}}_{\text{cont.}} + a_2 \underset{z \cdot z}{z^2} + a_3 \underset{z \cdot z^2}{z^3}$ etc.

And complex rational fcns are cont where denom $\neq 0$.

Def^n f is complex diff'ble ($\mathbb{C}$-diff) at z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. If it does, call

$f'(z_0)$. $\left[\text{or } \left. \frac{df}{dz} \right|_{z = z_0} \right]$

Fact: All rules about diff'tion $\mathbb{R} \to \mathbb{R}$ hold $\mathbb{C} \to \mathbb{C}$.

Ex: $(fg)' = f'g + fg'$

$$\frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} = \frac{f(z)g(z) - f(z_0)g(z) + f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= g(z)\cdot\underbrace{\frac{f(z)-f(z_0)}{z-z_0}} + f(z_0)\cdot\frac{g(z)-g(z_0)}{z-z_0}$$

$z \to z_0$: $$g(z_0)\cdot f'(z_0) + f(z_0)\cdot g'(z_0) \checkmark$$

Remark: Know f diff'ble at $z_0 \Rightarrow f$ cont at z_0:

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) + E(z)$$

where $E(z) \to 0$ as $z \to z_0$.

$$f(z) = f(z_0) + f'(z_0)\underbrace{(z-z_0)} + E(z)\underbrace{(z-z_0)}$$

$z \to z_0$: $$f(z) \to \underbrace{f(z_0) + f'(z_0)\cdot 0 + 0\cdot 0}_{f(z_0)} \checkmark$$

Def^n f is analytic on an open set Ω if it is $\mathbb{C}$-diff at each pt in Ω.

Fact: Complex polys are $\underbrace{\text{analytic on } \mathbb{C}}_{\text{entire fcns}}$.

Rational fcns too, where denom $\neq 0$.

EX: 1) $$z^2 = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + i\underbrace{(2xy)}_{v(x,y)}$$

2) $e^z = e^{x+iy} = (e^x \cos y) + i(e^x \sin y)$

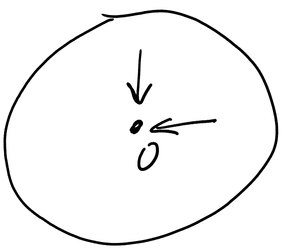
Is e^z $\mathbb{C}$-diff. Yes!

3) $\bar{z} = x - iy$ is <u>not</u> $\mathbb{C}$-diff'ble.

$$DQ = \frac{\bar{z} - \bar{0}}{z - 0} = \frac{\bar{z}}{z} \quad \leftarrow z \to 0$$

$re^{i\theta} \to 0$ exactly when $r \to 0$.

$$= \frac{\overline{re^{i\theta}}}{re^{i\theta}} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-i2\theta} \quad \leftarrow \text{what?}$$



different limits sliding into 0 along different radial directions.