Lesson 4 Complex power tools!

HWK 1 due in Gradescope by 11:30 pm tonight. Find link to GS in Brightspace.
HWK 2 on Home page. Due next Wed.

Geometric series:

$$\underbrace{1+z+z^2+\cdots+z^N}_{S_N(z)} = \frac{1}{1-z} - \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

$$\begin{array}{rl} & S_N(z) = 1+z+z^2+\cdots+z^N \\ - & zS_N(z) = \quad z+z^2+\cdots+z^N+z^{N+1} \\ \hline & 1 - z^{N+1} \end{array}$$

$$(1-z)S_N(z) = 1 - z^{N+1} \quad \checkmark$$

Fact: If $|z|<1$, then $S_N(z) \to \frac{1}{1-z}$ as $N\to\infty$.

Numerator estimate: $|z+w| \le |z|+|w|$

Denomenator estimate: $|z-w| \ge \big||z|-|w|\big|$ replace w by $-w$

also $|z+w| \ge \big||z|-|w|\big|$

$$\left|\frac{a+b}{c+d}\right| \le \frac{|a|+|b|}{\big||c|-|d|\big|}$$

Pf of fact: $\left|S_N(z) - \frac{1}{1-z}\right| = \left|\frac{-z^{N+1}}{1-z}\right|$

$$\le \frac{|z|^{N+1}}{\big|1-|z|\big|}$$

If $|z|<1$, then $\big|1-|z|\big| = 1-|z|$

$$\le \frac{|z|^{N+1}}{1-|z|} \quad \text{when } |z|<1.$$

$\to 0$ as $N \to \infty$ when $|z| < 1$.

<u>Note</u> $S_N(z)$ doesn't converge when $|z| \ge 1$ because the terms $|z^n| = |z|^n \ge 1$ <u>don't</u> go to zero as $n \to \infty$.

Power of the complex exponential

D'M: $\quad (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$

Famous formulas

$$e^{i\theta} = \cos\theta + i\sin\theta$$

← Euler's identity

$$+\quad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

← Future: Let $\theta \in \mathbb{C}$ to get complex cos, sin

Famous Fourier Series Fact

$$\underbrace{\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos N\theta}_{C_N} = \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$

$$C_N = \mathrm{Re}\Big[\underbrace{1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{iN\theta}}_{1 + (e^{i\theta}) + (e^{i\theta})^2 + \cdots + (e^{i\theta})^N}\Big] - \frac{1}{2}$$

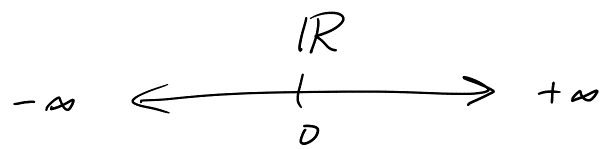
$$= \mathrm{Re}\left[\frac{1-(e^{i\theta})^{N+1}}{1-e^{i\theta}} \cdot \frac{-e^{-i\theta/2}}{-e^{-i\theta/2}}\right] - \frac{1}{2}$$

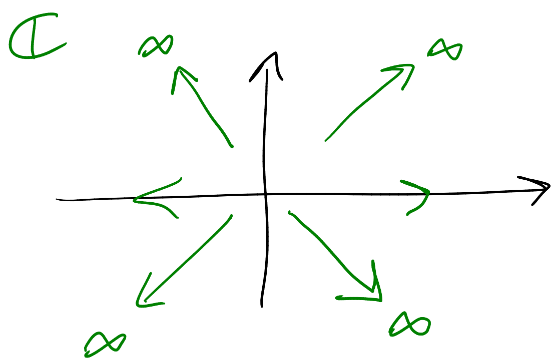
$$= \mathrm{Re}\left[\frac{e^{i(N+\frac{1}{2})\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \cdot \frac{\left(\frac{1}{2i}\right)}{\left(\frac{1}{2i}\right)}\right] - \frac{1}{2}$$

$$= \mathrm{Re}\left[\frac{-\frac{1}{2}i\left(\cos(N+\frac{1}{2})\theta + i\sin(N+\frac{1}{2}) - \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{\sin\theta/2}\right] - \frac{1}{2}$$

$$= \frac{\sin(N+\frac{1}{2})\theta}{2\sin\theta/2} \quad \checkmark$$

The "point at infinity"





Defⁿ: $\lim\limits_{z\to a} f(z) = \infty$

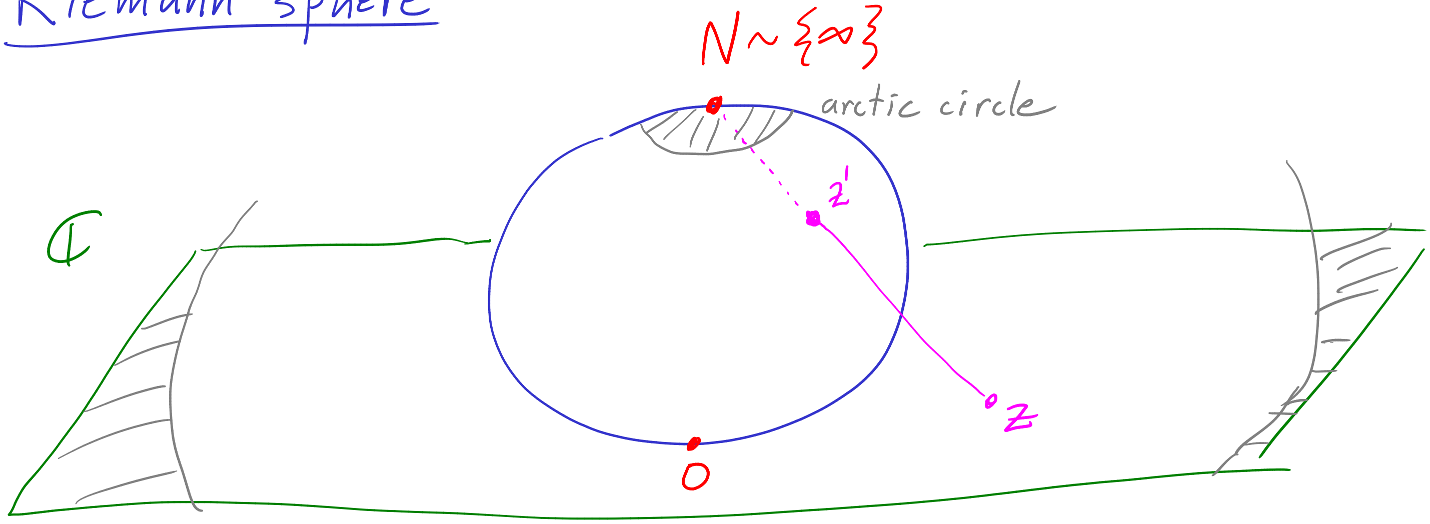
means $|f(z)| \to \infty$

as $z \to a$.

More precisely, given $M > 0$, there is a $\delta > 0$ such that $|f(z)| > M$ when $|z-a| < \delta$, $z \neq a$.

EX: $\lim\limits_{z\to a} \frac{1}{z-a} = \infty \qquad \lim\limits_{z\to a} \frac{1}{(z-a)^4} = \infty$

Riemann sphere

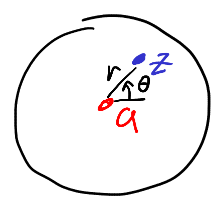


$[-\infty, \infty]$ "compactifies" $\mathbb{R}$

$$D_\varepsilon(\infty) = \{ z \in \mathbb{C} : |z| > \tfrac{1}{\varepsilon} \} \cup \{\infty\}$$ ← Disc about ∞ in $\hat{\mathbb{C}}$

Extended complex plane: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Last time. z^2 is $\mathbb{C}$-diff'ble.

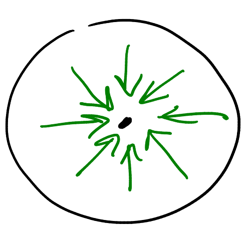


$\bar{z}$ is not: $DQ = \dfrac{\bar{z} - \bar{a}}{z - a}$

Write $z = a + re^{i\theta}$

$$= \frac{\overline{(a+re^{i\theta})} - \bar{a}}{(a+re^{i\theta}) - a}$$

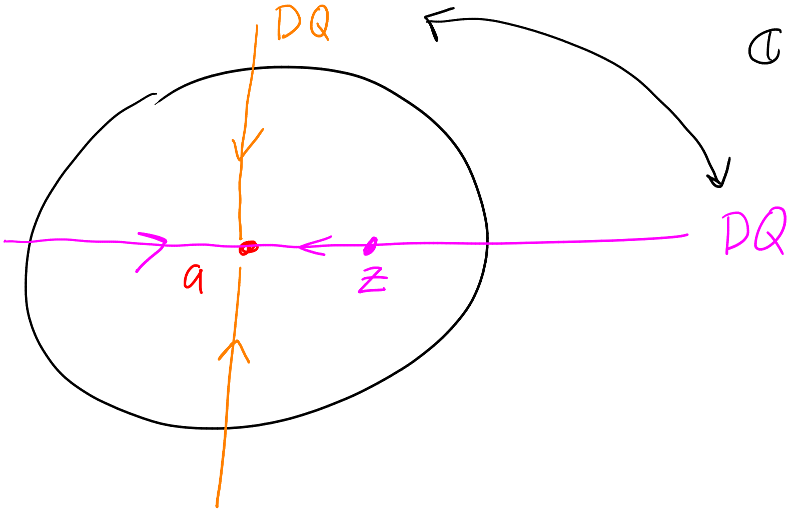
$$= \frac{r\,\overline{e^{i\theta}}}{r\,e^{i\theta}} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-i2\theta}$$



different limits as $\underbrace{r \to 0}_{z \to a}$ for diff θ.

Cauchy-Riemann eqns

$$DQ = \frac{f(z) - f(a)}{z - a}$$



$\mathbb{C}$- diff'ble $\Rightarrow$ these two DQ's have same limit.

Next time: equate two DQ's. Get C-R eqns.

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

C-R eqns
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$