Lesson 5 The Cauchy-Riemann equations

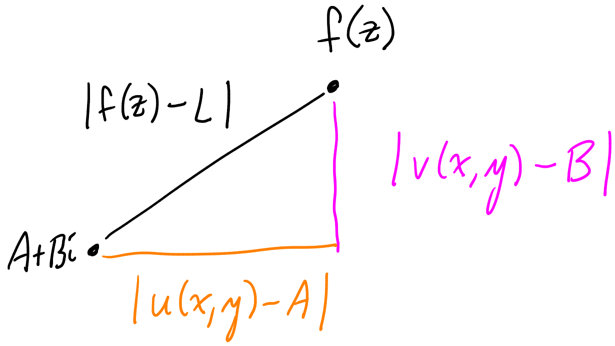
Read Chap 2

Lemma Write $f(x+iy) = u(x,y) + iv(x,y)$

$z = x+iy \qquad z_0 = x_0 + iy_0$

$$\lim_{z \to z_0} f(z) = \underbrace{A+Bi}_{L} \iff \begin{array}{l} \displaystyle\lim_{(x,y)\to(x_0,y_0)} u(x,y) = A \\ \displaystyle\lim_{(x,y)\to(x_0,y_0)} v(x,y) = B \end{array}$$

Why:

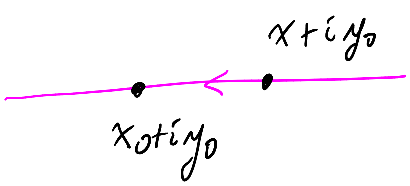


See lemma from

$\Rightarrow$ legs $\leq$ hypoteneus

$\Leftarrow$ Pyth Thm

Cauchy-Riemann equs: Suppose $f(z)$ is $\mathbb{C}$-diff at z_0.



$$DQ = \frac{f(z)-f(z_0)}{z-z_0}$$

$$= \frac{[u(x,y_0)+iv(x,y_0)] - [u(x_0,y_0)+iv(x_0,y_0)]}{(x+iy_0)-(x_0+iy_0)}$$

$$= \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\,\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}$$

Aha! If DQ $\to$ limit, then real and imag parts $\to$ limits.

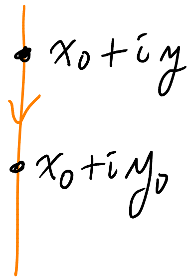
(by lemma). See first partials $\left.\dfrac{\partial u}{\partial x}\right|_{(x_0,y_0)}, \left.\dfrac{\partial v}{\partial x}\right|_{(x_0,y_0)}$

exist and

$$\boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)}$$

Red box #1

Next



$$DQ = \frac{f(x_0+iy) - f(x_0+iy_0)}{(x_0+iy) - (x_0+iy_0)}$$

$\frac{1}{i} = -i$

$$\frac{u(x_0,y) - u(x_0,y_0)}{i(y-y_0)} + \frac{i\left[v(x_0,y) - v(x_0,y_0)\right]}{i\,(y-y_0)}$$

$$\longrightarrow \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)$$

Lemma $\Rightarrow$ partials wrt y exist and

$$\boxed{f'(z_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)}$$

Red box #2

Red box #1 = Red box #2. Get C-R eqns!

<u>Thm</u> If $f = u+iv$ is $\mathbb{C}$-diff'ble at z_0, then all the first partials of u and v exist at (x_0,y_0) and they satisfy

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \quad \text{at } (x_0,y_0)$$

Furthermore $f'(z_0) = \begin{cases} u_x + i\, v_x \\ v_y - i\, u_y \end{cases}$ at (x_0, y_0)

EX: $f(z) = \overbrace{x - iy}^{\bar{z}}$: $u(x,y) = x$ $\quad v(x,y) = -y$

$$\frac{\partial u}{\partial x} = 1 \overset{?}{=} \frac{\partial v}{\partial y} = -1 \quad \text{No!}$$

$$\left[\ \frac{\partial u}{\partial y} = 0 \overset{?}{=} -\frac{\partial v}{\partial x} = 0 \quad \text{Yes, but who cares!}\ \right]$$

$f'(z)$ exists $\Rightarrow$ C-R eqns. They fail. So f' doesn't exist.

EX: $e^{x+iy} = \underbrace{\left(e^x \cos y\right)}_{u} + i \underbrace{\left(e^x \sin y\right)}_{v}$

C-R eqns hold! Is e^z $\mathbb{C}$-diff'ble?

We showed $\mathbb{C}$-diff'ble $\Rightarrow$ C-R eqns.

Is the reverse true. Yes! with some minimal further assumptions. [Need u, v to be C^1-smooth]

Thm $f(x+iy) = u(x,y) + i\, v(x,y)$.

If u and v are C^1-smooth on an open set Ω

[meaning u, v continuous, all first partials exist and are also continuous on Ω], and u, v satisfy the C-R eqns, then $f = u + iv$ is analytic on Ω. ($\mathbb{C}$-diff'ble on open Ω.)

Calculus lemma If $u(x,y)$ is C^1-smooth, then

$$u(x,y) = \underbrace{u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0)}_{\text{First order linear approx}} + \underbrace{R(x,y)}_{\text{remainder}}$$

where $$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Pf of Thm

$$\boxed{\begin{gathered} u^0 = u(x_0,y_0) \\ u_x^0 = \frac{\partial u}{\partial x}(x_0,y_0) \\ \vdots \end{gathered}}$$

$$DQ = \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \frac{(u + iv) - (u^0 + iv^0)}{z - z_0}$$

$$= \frac{\left[u_x^0(x-x_0) + \underbrace{u_y^0}_{=-v_x^0}(y-y_0)\right] + i\left[v_x^0(x-x_0) + \underbrace{v_y^0}_{=u_x^0}(y-y_0)\right]}{z - z_0} + \underbrace{\frac{R_u + iR_v}{z - z_0}}$$

Aha! $$= \frac{[u_x^0 + iv_x^0]\left((x-x_0) + i(y-y_0)\right)}{z - z_0} + R$$

$$= (\text{Red box } \#1) + \mathcal{R}$$

Finally, $|\mathcal{R}| \le \frac{|R_u| + |R_v|}{|z - z_0|} \le \frac{|R_u|}{\sqrt{\quad}} + \frac{|R_v|}{\sqrt{\quad}}$

$\longrightarrow 0$ as $z \to z_0$ by Calc lemma.

So $f'(z_0)$ exists and is equal to Red box #1.

Cor e^z is $\underbrace{\text{analytic on } \mathbb{C}}_{\text{entire}}$ and $\frac{d}{dz} e^z = e^z$.

Check $\frac{d}{dz} e^z = \text{Red box } \#1 = u_x + i v_x$

$$= \frac{\partial}{\partial x}(e^x \cos y) + i \frac{\partial}{\partial x}(e^x \sin y)$$

$$= e^x \cos y + i e^x \sin y = e^z \checkmark$$

Remark $\mathbb{R}^2 \to \mathbb{R}^2$ | $\mathbb{C} \to \mathbb{C}$

$(x,y) \mapsto (u(x,y), v(x,y))$ | $z \mapsto f(z)$

Jacobian matrix

$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \leftarrow \text{C-R eqns}$$

$f'(z)$ exists

$J : \mathbb{R}^2 \xrightarrow{\text{linear}} \mathbb{R}^2$

representing complex multiplication $\mathbb{C} \to \mathbb{C}$.

Think about doing C-R eqns twice.

Fact: If f analytic, then f' is analytic!

Consequence: Real and imag parts of analytic fcns are harmonic fcns!