# Lesson 6 Cauchy-Riemann equations, applications.

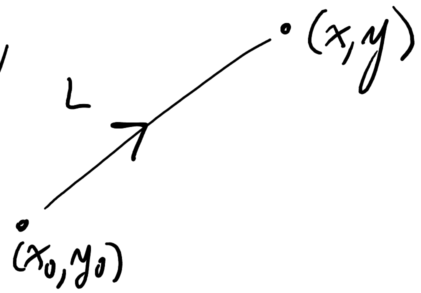
**Calculus lemma** Suppose $u(x,y)$ $C^1$-smooth. Then

$$u(x,y) = \underbrace{\overbrace{u(x_0,y_0)}^{u^0} + \overbrace{\frac{\partial u}{\partial x}(x_0,y_0)}^{u_x^0}(x-x_0) + \overbrace{\frac{\partial u}{\partial y}(x_0,y_0)}^{u_y^0}(y-y_0)}_{\text{First order (linear) Taylor approx to } u} + \underbrace{R(x,y)}_{\text{remainder}}$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Why




$$L: (x_0,y_0) + t(x-x_0, y-y_0)$$
$$0 \le t \le 1$$

$$u(x,y) - u(x_0,y_0) = \int_L \nabla u \cdot d\vec{s}$$

$$= \int_0^1 \left( \frac{\partial u}{\partial x}\underbrace{\left(x_0+t(x-x_0), y_0+t(y-y_0)\right)}, \frac{\partial u}{\partial y}\left(x_0+t(x-x_0), y_0+t(y-y_0)\right) \right) \cdot (x-x_0, y-y_0)\, dt$$

$$= \int_0^1 \frac{\partial u}{\partial x}\cdot(x-x_0) + \frac{\partial u}{\partial y}\cdot(y-y_0)\, dt$$

Hmmm. $\underbrace{\left(u_x^0\cdot(x-x_0)+u_y^0\cdot(y-y_0)\right)}_{\text{constant}} = \int_0^1 \left(u_x^0\cdot(x-x_0)+u_y^0\cdot(y-y_0)\right)dt$

$$R(x,y) = \int_0^1 (u_x-u_x^0)(x-x_0)+(u_y-u_y^0)(y-y_0)\,dt$$

$$|R(x,y)| \le \underbrace{\left(\max_L |u_x-u_x^0|\right)}_{\text{Make } < \frac{\varepsilon}{2}}\cdot|x-x_0| + \underbrace{\left(\max_L |u_y-u_y^0|\right)}_{\text{Make } < \frac{\varepsilon}{2}}\cdot|y-y_0|$$

Let $\varepsilon>0$.

$$\frac{|R(x,y)|}{\sqrt{\phantom{xx}}} \le \frac{\varepsilon}{2}\underbrace{\frac{|x-x_0|}{\sqrt{\phantom{xx}}}}_{\le 1} + \frac{\varepsilon}{2}\underbrace{\frac{|y-y_0|}{\sqrt{\phantom{xx}}}}_{\le\ [(\text{leg})\le(\text{hyp})]} < \varepsilon$$

Exciting consequence of C-R equs. Suppose $u,v$ $C^2$-smooth real valued fcns on $\mathbb{R}^2$ that satisfy the C-R equs (so $u+iv$ is analytic). Then $u$ and $v$ are harmonic!

[ $u$ harmonic: $\Delta u\equiv 0$ where $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. ]

Why

$$\begin{cases} u_x = v_y & (A)\\ u_y = -v_x & (B)\end{cases}$$

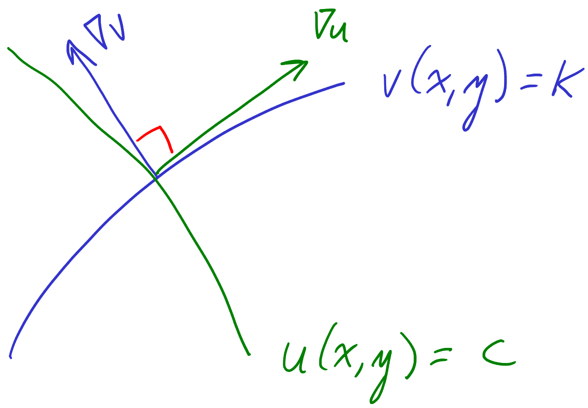
$$\begin{aligned} \tfrac{\partial}{\partial x}(A):&\quad u_{xx} = v_{xy}\\ +\ \tfrac{\partial}{\partial y}(B):&\quad u_{yy} = -v_{yx}\end{aligned}$$

$v$ $C^2$-smooth. So $v_{xy}=v_{yx}$

Aha! $u_{xx} + u_{yy} = v_{xy} - v_{yx} \equiv 0$. So harmonic.

Similarly $\frac{\partial}{\partial y}(A) - \frac{\partial}{\partial x}(B)$ shows $v$ harmonic.

Great fact: Level sets of $u$ & $v$ cross at right angles!




C-R eqns

$u_x = v_y$

$u_y = -v_x$

$\nabla u \cdot \nabla v$

$$\underbrace{(v_y, -v_x)}_{\text{from C-R eqns}} \cdot (v_x, v_y) = 0$$

Big fact: Real and imag parts of analytic fcns are $C^\infty$-smooth! So are harmonic fcns!

Question Given a $C^2$-smooth harmonic fcn $u$, can I find a $v$ so that $u+iv$ is analytic?

Yes! Want $v$ with $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ $\quad v_x = -u_y, \quad v_y = u_x$

$\boxed{v \text{ is called a harm conjugate for } u}$

Aha! Want $\nabla v = \vec{F}$ where $\vec{F} = (-u_y, u_x)$

In $\mathbb{R}^3$, need Curl $\vec{F} = 0$.

In $\mathbb{R}^2$: Think $\nabla\varphi = (\varphi_x, \varphi_y) = (F_1, F_2)$

Know $\varphi_{yx} = \varphi_{xy}$

$$\boxed{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}}$$

Can find $\varphi$ exactly when this holds

MA 261

In our case $F_1 = -u_y$, $F_2 = u_x$

Need $\frac{\partial}{\partial y}(-u_y) \stackrel{?}{=} \frac{\partial}{\partial x}(u_x)$ Yes. $\Delta u = 0$. ✓

EX: $u(x,y) = x^2 - y^2 + xy + 3$ Check: $\Delta u \equiv 0$ ✓

Find $v$ with
$$\begin{cases} v_x = -u_y = -\frac{\partial}{\partial y}(x^2 - y^2 + xy + 3) = 2y - x & (A) \\ v_y = u_x = 2x + y & (B) \end{cases}$$

(A) $v_x = 2y - x$

Step 1 Use (A): $v = \int 2y - x \, dx = 2xy - \frac{1}{2}x^2 + \underbrace{C(y)}_{\text{arb fcn of } y \text{ (the other var)}}$

treat $y$ like a const in $\int$ ("partial integral")

Got $\boxed{v = 2xy - \frac{1}{2}x^2 + C(y)}$ Done with (A).

Step 2 Use (B), but in a different way.

(B): $v_y = 2x + y$

Need $\dfrac{\partial}{\partial y}\boxed{2xy-\tfrac{1}{2}x^2+C(y)} \overset{\text{want}}{=} 2x+y$

$$2x-0+C'(y)=2x+y$$

Miracle: All $x$'s cancel!

$$\boxed{C'(y)=y}$$ (B) is used up.

So $C(y)=\int y\,dy=\tfrac{1}{2}y^2+K$

So harm conjugate $v$ is box: $\boxed{v=2xy-\tfrac{1}{2}x^2+\underbrace{\tfrac{1}{2}y^2+K}_{C(y)}}$

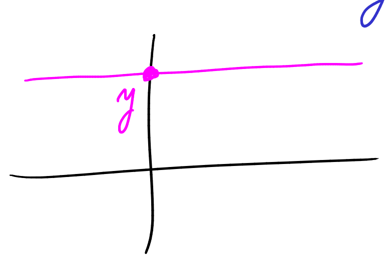
Hmmm. $u+iv$ is analytic. How to write in terms of $z$?

$$\begin{cases} z=x+iy \\ \bar{z}=x-iy \end{cases} \qquad \begin{cases} x=\dfrac{z+\bar{z}}{2} \\ y=\dfrac{z-\bar{z}}{2i} \end{cases}$$

plug into $u+iv$ formulas
Miracle: all $\bar{z}$'s will cancel out.

More about why $C(y)$ is an arb fcn of $y$.

Fact: If $\dfrac{\partial u}{\partial x}\equiv 0$:




← $u$ const along horiz lines.

So $u$ only depends on $y$. Conversely $\dfrac{\partial}{\partial x}[C(y)]\equiv 0$.

Our $v$: $\dfrac{\partial v}{\partial x}=2y-x$

$v=2xy-\tfrac{1}{2}x^2$ satisfies this. So

$\dfrac{\partial}{\partial x}\left[v-\left(2xy-\tfrac{1}{2}x^2\right)\right]\equiv 0$ and thing [ ] = fcn of $y$ alone.