Lesson 7 Analytic functions, complex exponential

EX $u(x,y) = x^3 - 3xy^2 + y$ is harmonic. [Check: $\Delta u \equiv 0$]

Get harm conjugate: v such that $u+iv$ is analytic.

[need u C^2-smooth, harmonic today. Fact: harmonic fcns C^∞-smooth!]

Want
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \longrightarrow \begin{cases} v_x = -u_y \\ v_y = u_x \end{cases} \qquad \nabla v = (-u_y, u_x)$$

Need
$$\begin{cases} v_x = -\frac{\partial}{\partial y}\left(x^3 - 3xy^2 + y\right) = 0 + 6xy - 1 & (A) \\ v_y = \frac{\partial}{\partial x}\left(x^3 - 3xy^2 + y\right) = 3x^2 - 3y^2 + 0 & (B) \end{cases}$$

$$\begin{cases} v_x = 6xy - 1 & (A) \\ v_y = 3x^2 - 3y^2 & (B) \end{cases}$$

Step 1: Use (A): $v = \int 6xy - 1 \, dx = \boxed{3x^2y - x + \underbrace{C(y)}_{\text{arb fcn of } y}}$

Step 2: Use (B) in a different way

$$\frac{\partial}{\partial y}\left[3x^2y - x + C(y)\right] \overset{\text{want}}{=} 3x^2 - 3y^2$$

$$3x^2 - 0 + C'(y) \overset{\text{want}}{=} 3x^2 - 3y^2$$

All x's cancel!

$$\boxed{C'(y) = -3y^2}$$

So $C(y) = \int -3y^2\,dy = -y^3 + K$

$x = \frac{z+\bar{z}}{2}$

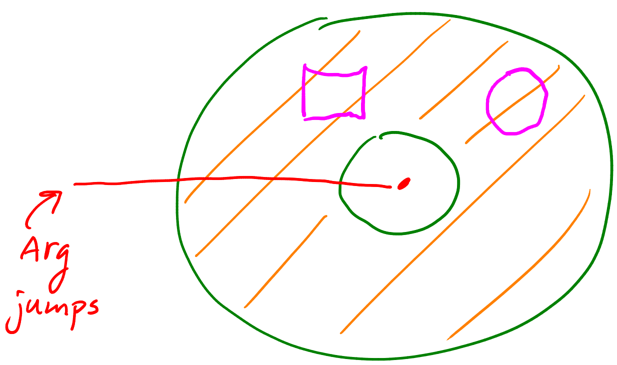
$y = \frac{z-\bar{z}}{2i}$

Get $\boxed{v = 3x^2y - x + (-y^3 + K)}$

Fact: Harm conj are uniq up to $+K$ on a domain (if they exist).

Danger! If a domain has holes, harm conj might not exist.

EX: $u = \mathrm{Ln}\sqrt{x^2+y^2} = \mathrm{Ln}|z|$. On a disc or a rectangle, get $v = \mathrm{Arg}\, z + K = \mathrm{Tan}^{-1}\frac{y}{x} + K$



$v = \mathrm{Arg}\, z$ ← Principal branch arg

on annulus minus $\mathbb{R}^-$.

Can't hook it up to be cont along $\mathbb{R}^-$.

Basic facts: 1) Analytic fcns must be continuous

Pf: Last week.

2) If f analytic on a domain Ω and $f'(z) \equiv 0$, then f is constant on Ω.

Pf of (2): $f(x+iy) = u + iv$

C-R eqns: $f'(z) = \begin{cases} u_x + i v_x & \text{Red box \#1} \\ v_y - i u_y & \#2 \end{cases}$

Aha! See $\nabla u \equiv 0$, $\nabla v \equiv 0$ when $f' \equiv 0$.

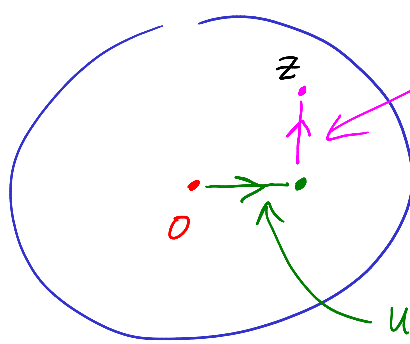
Know: u, v continuous, first partials <u>exist</u>.

Enough to see u, v are const.

<u>Freshman calculus fact</u> If $h(t)$ is continuous on $[a,b]$ and $h'(t)$ exists on (a,b) and $\equiv 0$ there, then $h(t) \equiv$ const on $[a,b]$.

<u>Pf</u>: MVT

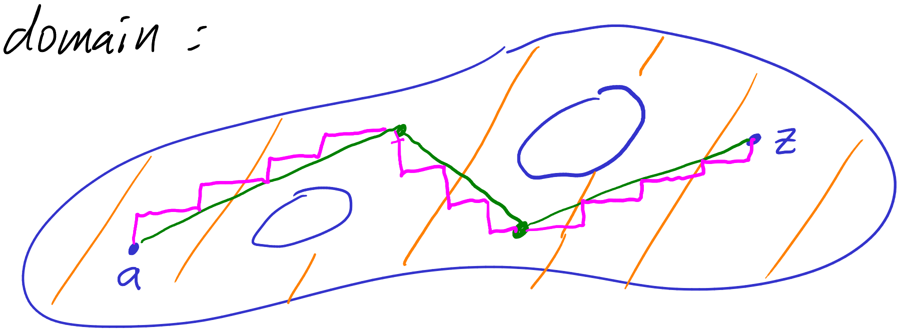
(2) on a disc:



$u_y, v_y \equiv 0$ along vert.
u, v const along vert

$u_x \equiv 0$, $v_x \equiv 0$ along horiz.
Fresh calc $\Rightarrow$ u, v const along horiz.

(2) on a domain:



Connect a to z by path. Approx by piecewise linear. Zig-zag along horiz and vert lines as in disc case.

See $u(a) = u(z)$.

Thm: Complex chain rule: $h(z)=f(g(z))$. $w_0=g(z_0)$. If g is $\mathbb{C}$-diff'ble at z_0 and f is $\mathbb{C}$-diff'ble at w_0, then h is $\mathbb{C}$-diff'ble at z_0 and

$$h'(z_0)=f'(g(z_0))\,g'(z_0).$$

Pf: $\dfrac{f(w)-f(w_0)}{w-w_0}=f'(w_0)+E(w)$ where $E(w)\to 0$ as $w\to w_0$

Define $E(w_0)=0$ to make it cont at w_0.

$$f(w)-f(w_0)=f'(w_0)(w-w_0)+E(w)(w-w_0)$$ ← holds at w_0 too.

Let $w=g(z)$, $w_0=g(z_0)$. Divide eqn by $z-z_0$:

$$DQ=\frac{f(g(z))-f(g(z_0))}{z-z_0}=f'(g(z_0))\underbrace{\frac{g(z)-g(z_0)}{z-z_0}}+\underbrace{E(g(z))}\frac{(g(z)-g(z_0))}{z-z_0}$$

$$z\to z_0:\qquad DQ\to f'(g(z_0))\cdot g'(z_0)+0\cdot g'(z_0)$$

because $g'(z_0)$ exists $\Rightarrow g$ cont at z_0. So $g(z)\to w_0=g(z_0)$.

Get chain rule.

Fun thing to do: Instead, use $\mathbb{R}^2\to\mathbb{R}^2$ chain plus C-R eqns.

Another fun thing to do. Cook up complex exponential fcn $E(z)$ such that $E'(z)=E(z)$ and $E(0)=1$.

Write $E(x+iy) = u + iv$

$$E'(z) = \begin{cases} u_x + i v_x & \text{(A)} \\ v_y - i u_y & \text{(B)} \end{cases} \overset{\text{want}}{=} u + iv$$

(A) : $u_x = u$ $\qquad$ $v_x = v$

$\boxed{u = C(y)e^x}$ $\qquad$ $\boxed{v = K(y)e^x}$

(B) : $v_y = u$ $\qquad$ $-u_y = v$

$\boxed{K'(y) = C(y)}$ $\qquad$ $\boxed{-C'(y) = K(y)}$

(1) $\qquad$ (2)

(1') $\quad K''(y) = C'(y) = -K(y)$ from (2)

$K''(y) + K(y) = 0$ $\qquad$ $K(y) = A\cos y + B\sin y$

(2') plus (1) : Similar. Get $C''(y) + C(y) = 0$.

Final thing. Get A's, Bs from $E(0) = 1$.

Get $E(z) = (e^x \cos y) + i\,(e^x \sin y)$. Ha!