Lesson 8 More about the complex exponential

Today we write $E(x+iy) = (e^x \cos y) + i(e^x \sin y)$

Soon we will write e^z instead of $E(z)$.

Properties of $E(z)$

1) $E(0) = 1$ ✓
2) $E'(z) = E(z)$ ✓
3) $E(z) \neq 0$ for all z ✓
4) $E(z+w) = E(z)E(w)$ ✓ ← $E(z)$ is a group homomorphism between the additive group of complex #s on $\mathbb{C}$ and multiplicative group on $\mathbb{C} - \{0\}$.
5) $E(-z) = 1/E(z)$ ✓
6) $E(z) = 1 \iff z = n(2\pi i)$, $n \in \mathbb{Z}$

Lemma If f is analytic on $\mathbb{C}$ (or $D_R(0)$) and $f'(z) = k f(z)$, then $f(z) = c\,E(kz)$ where $c = f(0)$.

Why

$$f' - kf = 0$$

← MA 262: Integrating factor is e^{-kz}

$$\underbrace{E(-kz) f'(z) - k E(-kz) f(z)}_{} \equiv 0$$

Hmmm.

$\frac{d}{dz}(E(-kz)) =$

$E'(-kz) \frac{d}{dz}(-kz)$

$E(kz)(-k)$

$$\frac{d}{dz}\left[E(-kz) f(z)\right] \equiv 0$$

So $\boxed{E(-kz) f(z) = c}$, a const.

Plug in $z=0$. See $c = f(0)$.

Stop the proof!

Sneaky idea. Restart proof using $f(z)=E(kz)$.

Since $\frac{d}{dz}E(kz) = k\,E(kz)$ by chain rule, this f satisfies the hyp. Aha! Get red box $\boxed{E(-kz)E(kz)=c}$

$z=0$: $c=1$.

Get $\underbrace{E(-kz)E(kz)}_{\text{so neither is } 0} = 1$

and $E(-kz) = 1/E(kz)$ continue with pf

Previous red box $\underbrace{E(-kz)}_{1/E(kz)} f(z) = c$

Get $f(z) = f(0)\,E(kz)$ ✓

Very cool thing Let $f(z) = E(z+w)$, $w \in \mathbb{C}$ (constant).

Chain rule: $\frac{d}{dz}E(z+w) = E'(z+w)\frac{d}{dz}(z+w)$

$= E(z+w)\cdot 1$

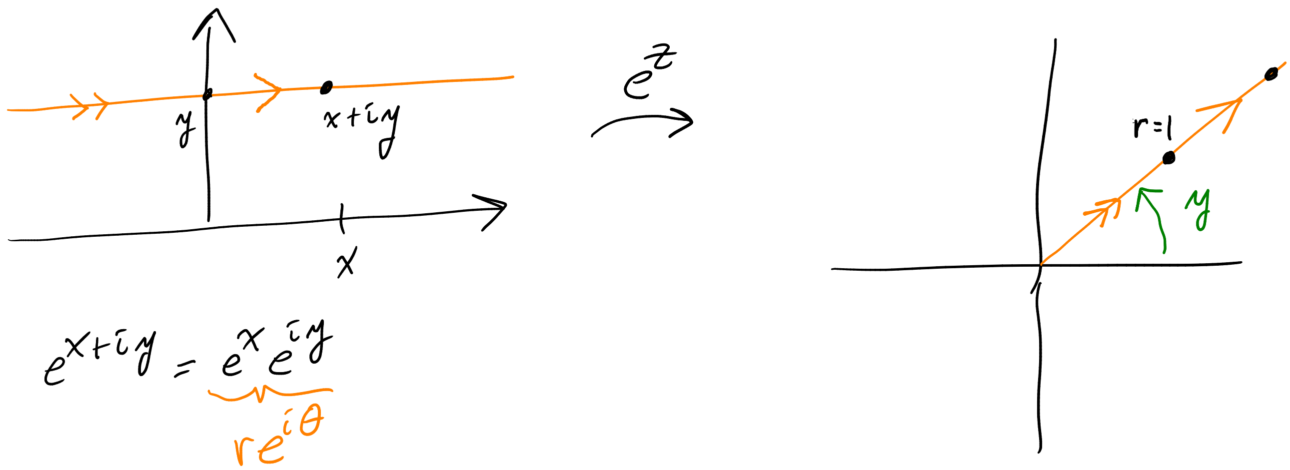
This $f(z)=E(z+w)$ satisfies hyp: $f'(z) = 1\cdot f(z)$.

Lemma $\Longrightarrow$ $f(z) = f(0)\,E(1\cdot z)$

$E(z+w) = E(0+w)\,E(z)$ ✓

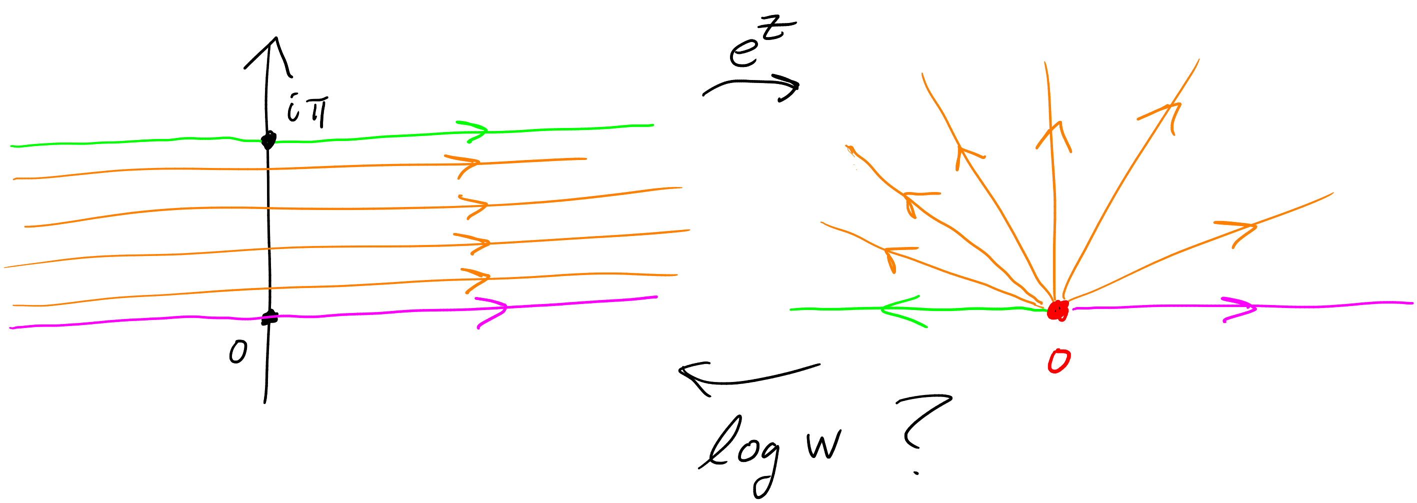
Wow! Can deduce truckloads of trig identities without looking at a single triangle!

Graph of e^z :

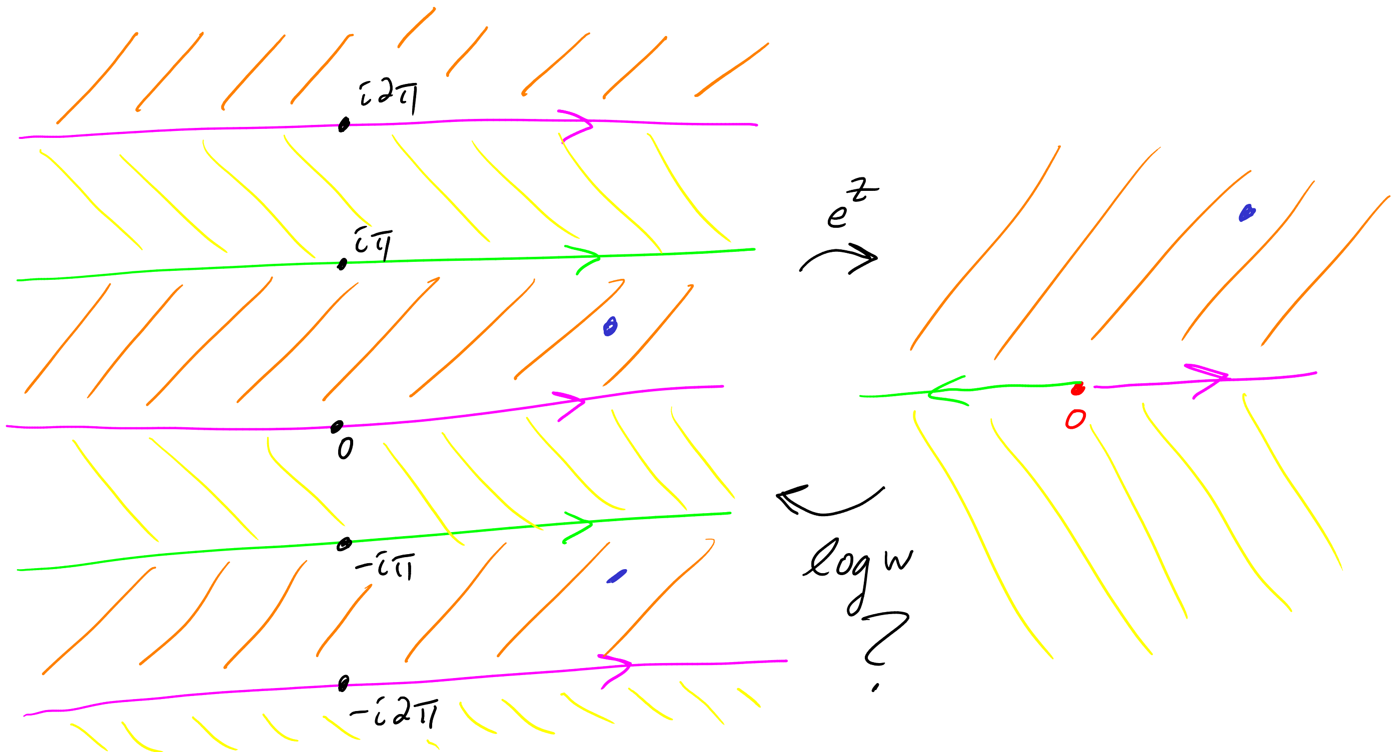


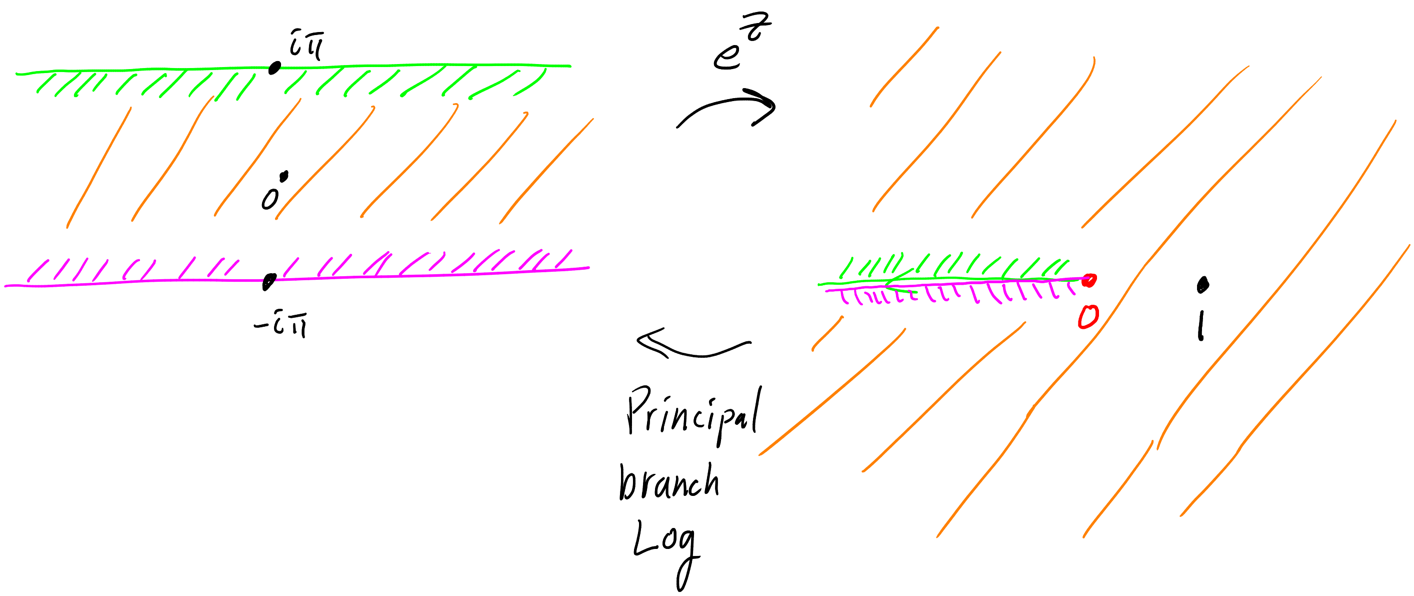
$$e^{x+iy} = \underbrace{e^x e^{iy}}_{re^{i\theta}}$$

$$-\infty < x < \infty$$
$$0 < e^x < \infty$$



Fact: e^z maps strip $\{z : 0 < \operatorname{Im} z < \pi\}$ 1-1 onto the upper half plane (UHP) $\{z : \operatorname{Im} z > 0\}$.





Think $\quad w = e^z = e^{x+iy} = e^x e^{iy}$

$= r e^{i\theta}$ where $r = |w| = e^x \leftarrow x = \mathrm{Ln}|w|$

$\theta = y \in \arg z$

$$z = \log w = (\mathrm{Ln}|w|) + i\theta, \quad \theta \in \arg z.$$

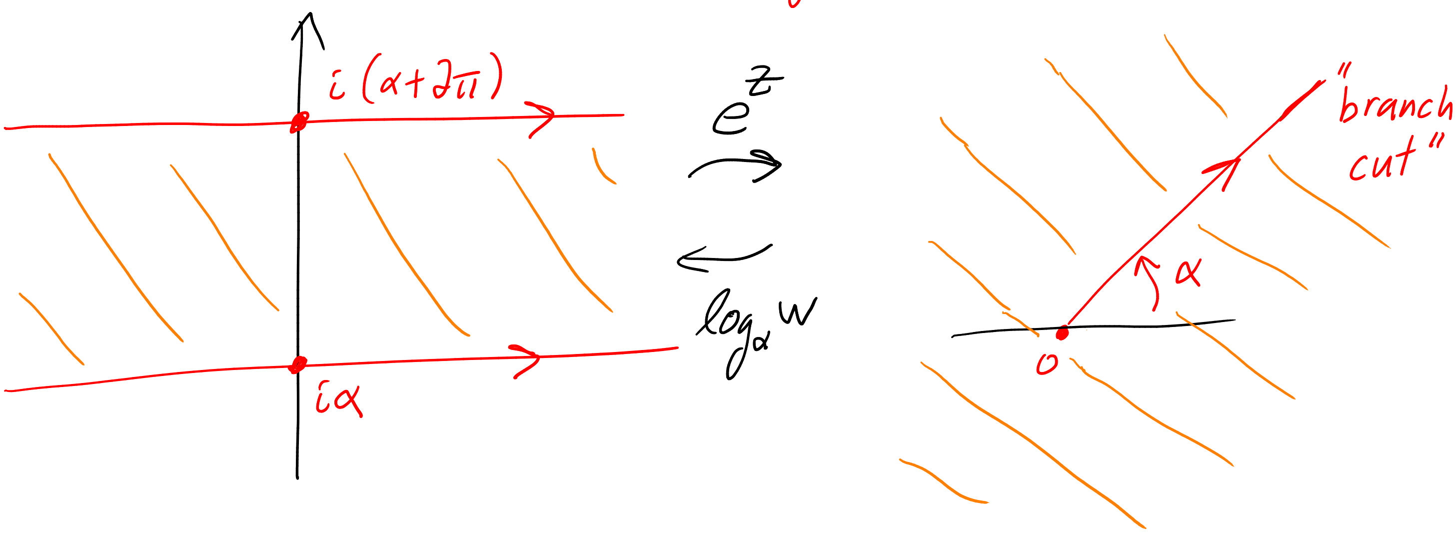
My def^n: Ln is reserved for the real natural log.

$$\log w = \{ z : e^z = w \} = \{ \operatorname{Ln}|w| + i\theta : \theta \in \arg z \}$$

$$= \{ \operatorname{Ln}|w| + i\theta : \theta = \operatorname{Arg} z + 2\pi n,\ n \in \mathbb{Z} \}$$

$\operatorname{Log} w$ = Principal branch of complex log fcn

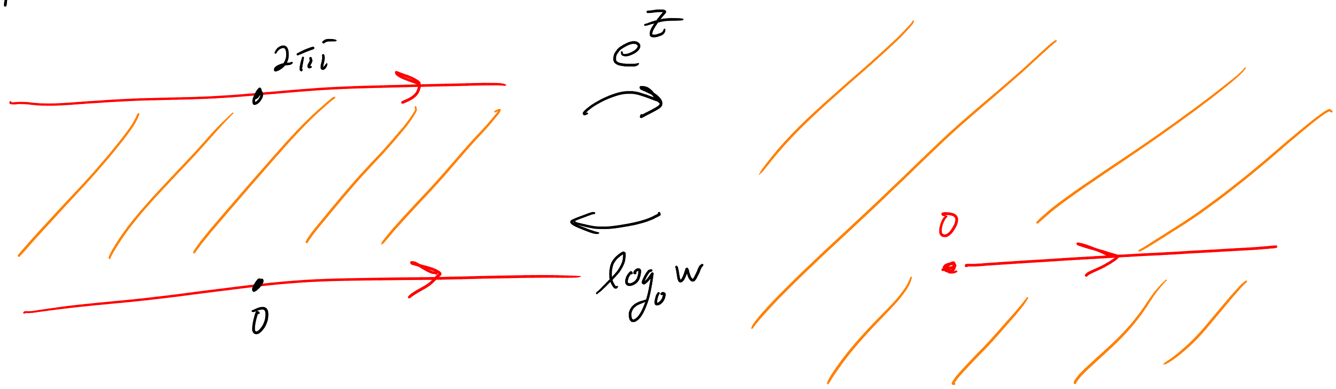
$$= \operatorname{Ln}|w| + i \underbrace{\operatorname{Arg} w}_{-\pi < \operatorname{Arg} w \leq \pi}$$



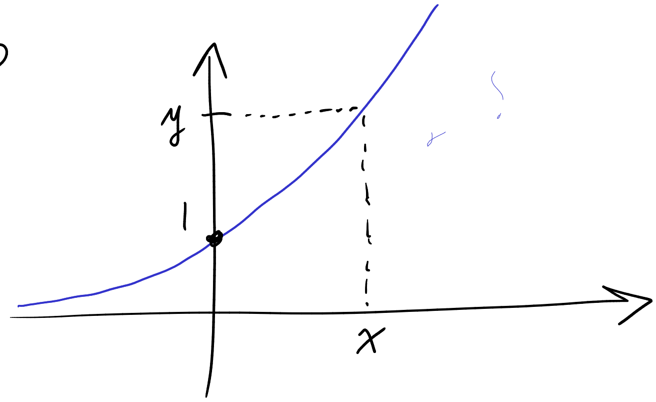
My def^n: $\log_\alpha w = \operatorname{Ln}|w| + i\theta$, $\theta \in \arg w$
$\alpha < \theta < \alpha + 2\pi$

"Branch of log"

Popular branch:



Real exp



Exactly one x such that $e^x = y$.

$$x = \ln y$$