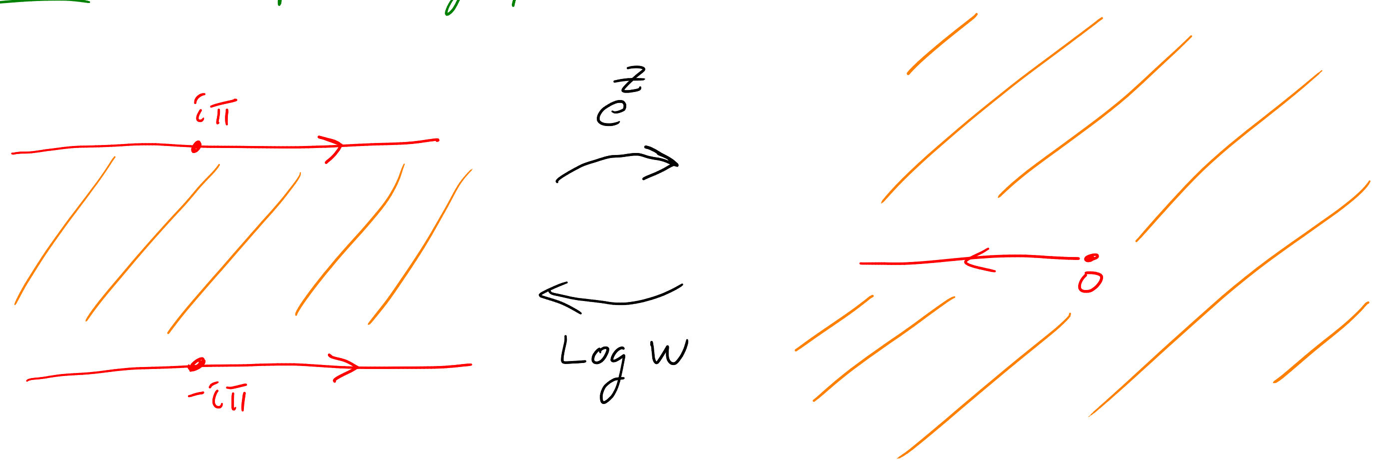
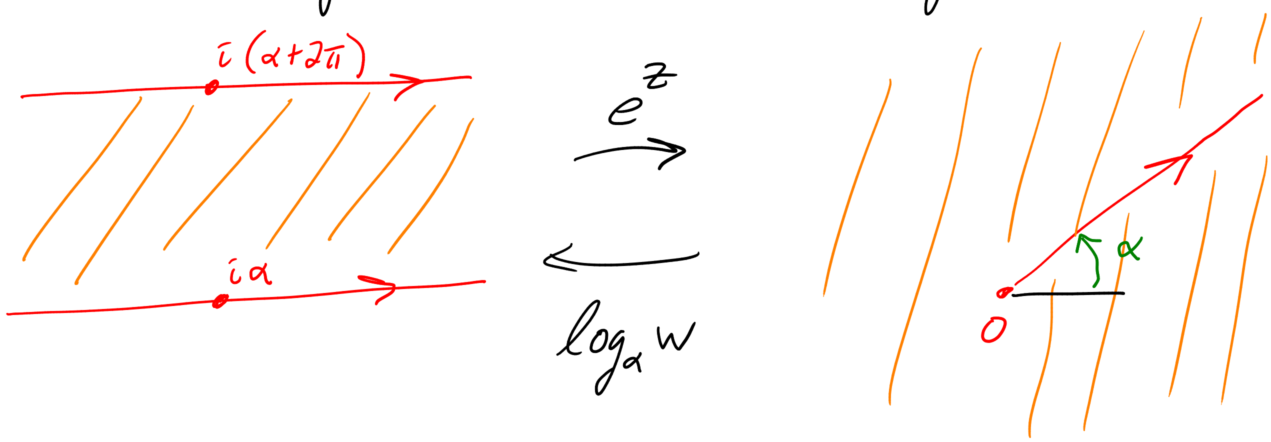
Lesson 9 Complex logarythm(s)



Princ. branch: $\text{Log}\, w = \ln|w| + i\theta$, $\theta \in \arg w$: $-\pi < \theta \leq \pi$



$$\log_\alpha w = \ln|w| + i\theta, \quad \theta \in \arg w \quad \alpha < \theta < \alpha + 2\pi$$

Cauchy-Riemann eqns in polar form

$$f(re^{i\theta}) = \bar{U}(r,\theta) + i\bar{V}(r,\theta)$$

If U and V are $\mathcal{C}'$-smooth on a polar rectange $(r \neq 0)$, and

$$\begin{cases} \dfrac{\partial U}{\partial r} = \dfrac{1}{r}\dfrac{\partial V}{\partial \theta} \\[2ex] \dfrac{\partial V}{\partial r} = -\dfrac{1}{r}\dfrac{\partial U}{\partial \theta} \end{cases}$$ polar C-R eqns

then f is analytic in the polar rec.

EX: $\log_\alpha w = \operatorname{Ln}|w| + i\theta \qquad w = re^{i\theta}$

$$\log_\alpha re^{i\theta} = \underbrace{\operatorname{Ln} r}_{U} + i\underbrace{\theta}_{V} \qquad \alpha < \theta < \alpha + 2\pi$$

Polar C-R eqns ✓ easy. U, V $\mathcal{C}'$-smooth ✓

Prob: Find $\dfrac{d}{dz}\left(\log_\alpha z\right)$

Hmmm. (1) $e^{\log_\alpha z} = z$ ← always a true =

(2) $\log_\alpha e^{z} = z + i2\pi n$ for $n \in \mathbb{Z}$ ← danger!

Diff (1) using chain rule:

$$1 = \frac{d}{dz}\Big(\underbrace{e^{\log_\alpha z}}_{z}\Big) = \frac{d}{dz} E(\log_\alpha z)$$

$$= E'(\log_\alpha z)\frac{d}{dz}\log_\alpha z$$

$$= \underbrace{E(\log_\alpha z)}_{e^{\log_\alpha z} = z}\frac{d}{dz}\log_\alpha z$$

Aha! $\boxed{\frac{d}{dz}\left(\log_\alpha z\right) = \frac{1}{z}}$

<u>Warning!</u> $\log_\alpha z_1 z_2 \stackrel{?}{=} \log_\alpha z_1 + \log_\alpha z_2$

Maybe not! True: $\log_\alpha z_1 z_2 = \log_\alpha z_1 + \log_\alpha z_2 + i2\pi n$

for some $n \in \mathbb{Z}$.

[$\log z_1 z_2 = \log z_1 + \log z_2$ as <u>sets</u>. $\log z = \{w : e^w = z\}$]

$\operatorname{Log} z_1 z_2 = \operatorname{Log} z_1 + \operatorname{Log} z_2$ when z_1, z_2 in first quadrant

Fantastic thing about classic complex fcns, e^z and its inverse, log fcn. In $\mathbb{R}$, entire zoo of fcns:

e^x, $\operatorname{Ln} x$, $\sin x$, $\cos x$, $\tan x$, ...

$\sinh x$, $\cosh x$, $\tanh x$, ...

Algebraic fcns $\sqrt{x}$, $\sqrt[n]{x}$ ← from solving polynomial eqns.

Called "Elementary fcns"

Claim: Zoo is two animals in $\mathbb{C}$!

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Let $\theta = z$ to get complex version

Hmmm. $\cos^{-1} w = z$

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Mult by e^{iz}:

$$2w = e^{iz} + \left(\frac{1}{e^{iz}}\right)$$

$$0 = \left(e^{iz}\right)^2 - (2w)\left(e^{iz}\right) + \underline{1}$$

Quadratic eqn in e^{iz}!

pick $\pm\sqrt{}$

$$e^{iz} = \frac{2w + \sqrt{4w^2 - 4\cdot1\cdot1}}{2\cdot1}$$

$$e^{iz} = w + \sqrt{w^2-1}$$

$$iz = \log\left(w + \sqrt{w^2-1}\right)$$

$$z = -i\log\left(w + \sqrt{w^2-1}\right) = \cos^{-1} w$$

Famous problem: $\int e^{-x^2}dx$ cannot be expressed as an elementary fcn.

Liouville's theorem: In $\mathbb{C}$: Elementary fcns are "generated" by e^z, its inverse, and algebraic fcns.

Remark: $\sinh z$, etc and $\sinh^{-1} z$, ... etc can also be expressed in terms of e^z and $\log z$ plus algebraic fcns like $\sqrt{z}$, polynomials.

Polar C-R eqns: $f(x+iy) = u(x,y) + i\,v(x,y)$

$$f(re^{i\theta}) = \bar{U}(r,\theta) + i\,\bar{V}(r,\theta)$$

$$\bar{U}(r,\theta) = u(r\cos\theta, r\sin\theta)$$
$$\bar{V}(r,\theta) = v(r\cos\theta, r\sin\theta)$$

$$\bar{U}_r = u_x\cos\theta + u_y\sin\theta$$
$$\bar{U}_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$\frac{1}{r}\bar{U}_\theta = u_x(-\sin\theta) + u_y\cos\theta$$

$\bar{V}$ derivatives similar

$$(*)\quad \begin{bmatrix} \bar{U}_r & \bar{V}_r \\ \frac{1}{r}\bar{U}_\theta & \frac{1}{r}\bar{V}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$$

Polar C-R eqns $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ ↑ has same appearance $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ Cartesian C-R eqns $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

Easy fact: Multiplying $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} A & -B \\ B & A \end{bmatrix} =$ Same kind of matrix.

So Cartesian C-R eqns $\Rightarrow$ Polar.

Hmmm. $$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\underbrace{\cos^2\theta+\sin^2\theta}_{1}} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{same kind of matrix!}}$$

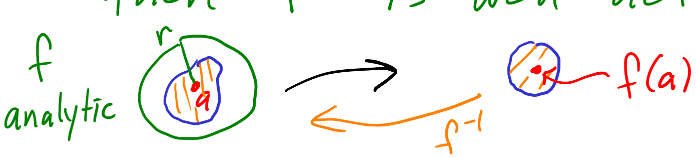
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Mult by $\begin{bmatrix} \cos & \sin \\ -\sin & \cos \end{bmatrix}^{-1}$ on left:

See Polar C-R eqns $\Rightarrow$ Cartesian

Next time: Complex inverse fcn thm.

If is analytic on $D_r(a)$ and $f'(a) \neq 0$, then f^{-1} is well defined near $f(a)$.



Review $\mathbb{R}^2 \to \mathbb{R}^2$ inv fcn thm.