Lesson 10 Complex inverse fuction theorem, i^i, Sec. 3.4

HWK 3 due tonight, HWK 4 posted, due next Wed.

Cmplx inv fcn thm If is analytic on $D_r(a)$ and $f'(a) \neq 0$, then f^{-1} is well defined in a disc $D_\varepsilon(f(a))$ and is analytic there. And

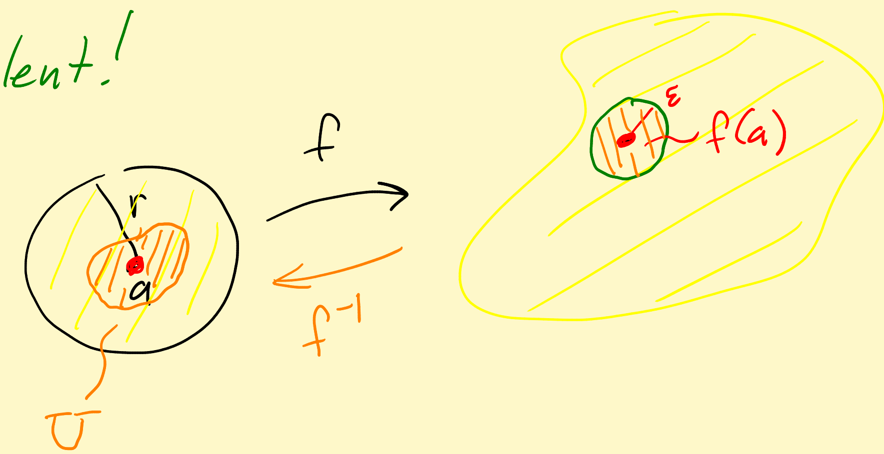
$$\frac{d}{dw}\left(f^{-1}(w)\right) = \frac{1}{f'(f^{-1}(w))}$$

MA 425: Define $f = u + iv$ analytic to mean u, v $\mathcal{C}^1$-smooth and satisfy C-R eqns. So $DQ = \frac{f(z)-f(a)}{z-a} \to f'(a)$.

MA 530: Analytic means only that $DQ \to f'(a)$ on open set.

Remark Equivalent!

$\mathbb{C}$ Inv fcn thm



$f: U \xrightarrow[\text{onto}]{1\text{-}1} D_\varepsilon(f(a))$ $\qquad$ f^{-1} analytic on $D_\varepsilon(f(a))$

Why: Follows from $\mathbb{R}^2 \to \mathbb{R}^2$ inverse fcn thm

$$f(x+iy) = u(x,y) + iv(x,y)$$

$$(x,y) \longmapsto (u(x,y), v(x,y))$$

Jacobian matrix $\quad J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \underset{\text{C-R eqns}}{=} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

$$\det J_f = \det \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = (u_x)^2 + (v_x)^2$$

$$= |\underbrace{u_x + i v_x}_{= f'}|^2 \neq 0$$

red box #1

So f^{-1} exists and is C^1-smooth on a small disc about $f(a)$.

$$J_{f^{-1}} = J_f^{-1} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{-1} = \underbrace{\frac{1}{A^2+B^2} \begin{bmatrix} A & B \\ -B & A \end{bmatrix}}$$

Aha! See C-R eqns!

Conclude f^{-1} is analytic.

Formula is now just the $\mathbb{C}$ chain rule:

$$f^{-1}(f(z)) = z$$

So $\quad (f^{-1})'(\underbrace{f(z)}_{w})\, f'(z) = 1$

$z = f^{-1}(w)$.

Remark: Could have used this to see $\log w$ is analytic because

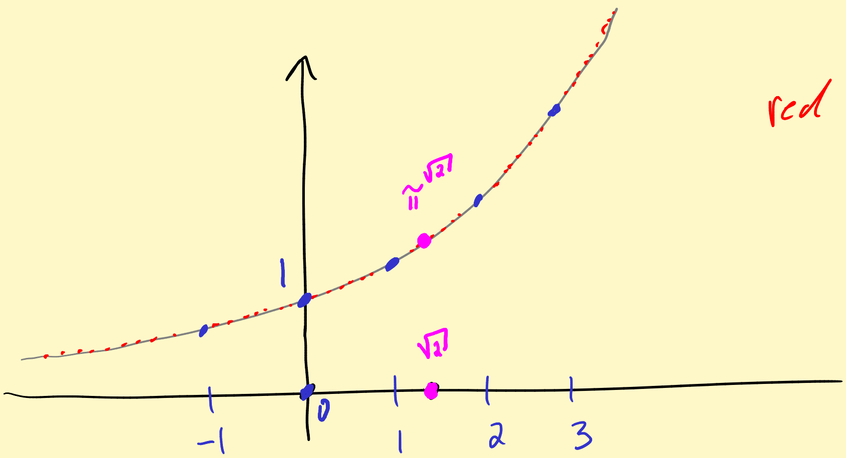
$\frac{d}{dz}(e^z) = e^z$ ← never zero.

Problem: What is i^i ?

Hmmm. $\pi^n = \underbrace{\pi \cdot \pi \cdots \pi}_{n\text{-times}}$

$$\pi^{p/q} = \sqrt[q]{\pi^p}$$ ← one real value. the other $q-1$ are complex $\notin \mathbb{R}$.

$$\pi^{\sqrt{2}} = e^{\operatorname{Ln} \pi^{\sqrt{2}}} = e^{\sqrt{2}\,\operatorname{Ln} \pi}$$



red dots $\pi^{p/q} \in \mathbb{Q}$

Complex version: $z^w = e^{w \log z} = \{ e^{w \cdot \zeta} : \zeta \in \log z \}$

∞ many choices of $\log z$!

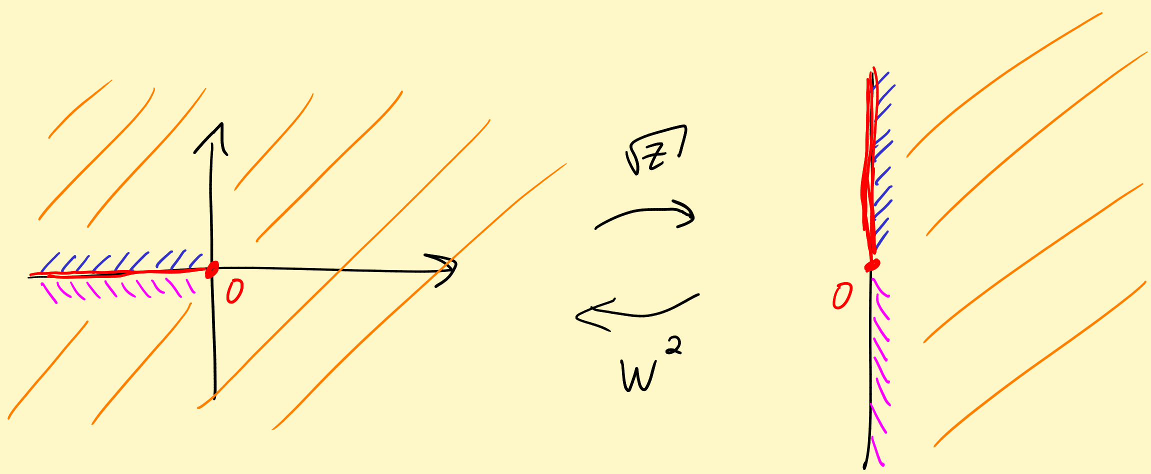
<u>Def^n</u> Principal branch of $z^w = e^{w \operatorname{Log} z}$ ← princ branch log

Principal square root $\sqrt{z} = e^{\frac{1}{2} \operatorname{Log} z}$

$$= e^{\frac{1}{2}(\mathrm{Ln}|z| + i\,\mathrm{Arg}\, z)} \qquad \theta = \mathrm{Arg}\, z,\ -\pi < \theta \le \pi$$

$$= \underbrace{e^{\frac{1}{2}\mathrm{Ln}|z|}}_{e^{\mathrm{Ln}|z|^{1/2}} = |z|^{1/2}}\ e^{i\theta/2}$$

$$\sqrt{r e^{i\theta}} = \sqrt{r}\, e^{i\theta/2} \qquad -\pi < \theta \le \pi$$



Easy facts z^n only has one value:

$$z^n = e^{n \log z} = e^{n[\mathrm{Ln}|z| + i(\mathrm{Arg}\, z + 2\pi k)]}$$

$$= e^{n\,\mathrm{Ln}|z|} \cdot e^{n i\,\mathrm{Arg}\, z} \cdot \underbrace{e^{i 2nk\pi}}_{=1}$$

$$= \left(e^{\mathrm{Ln}|z| + i\,\mathrm{Arg}\, z}\right)^n = \underbrace{z \cdot z \cdots z}_{n\text{-times}}$$

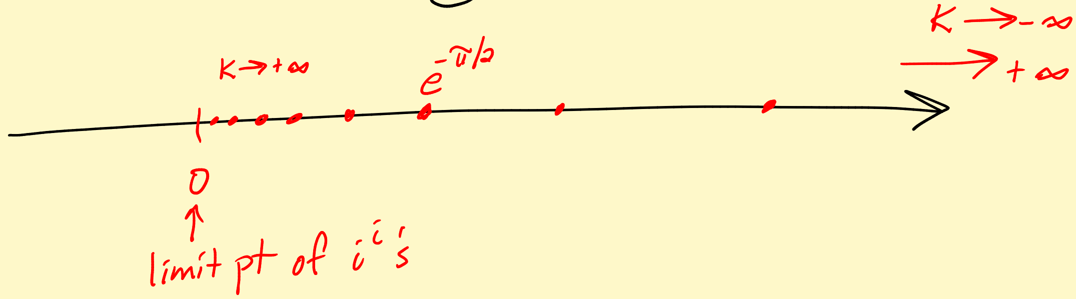
Also $z^{p/q}$ has q values in $\mathbb{C}$.

Princ value $i^i = e^{i\,\mathrm{Log}\,i} = e^{i(\underbrace{\ln|i|}_{=0} + i\frac{\pi}{2})} = e^{-\pi/2} \in \mathbb{R}!$

All i^i's: $i^i = e^{i\log i} = \left\{ e^{i\left(\ln|i| + i\left(\frac{\pi}{2}+2\pi k\right)\right)} \right\}$

$$= e^{-\left(\frac{\pi}{2}+2\pi k\right)}$$

Whoa!



3.4 Washers, wedges, walls

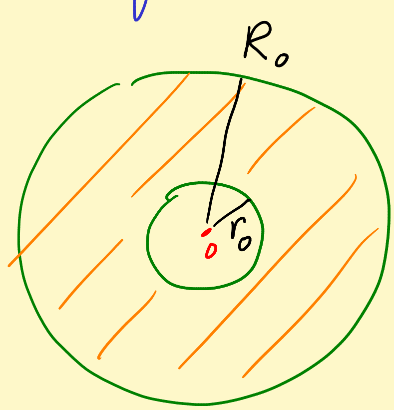
Heat eqn: $\frac{\partial u}{\partial t} = c\Delta u$

At "steady state", $\frac{\partial u}{\partial t} \equiv 0$.

$\Delta u \equiv 0$ solves temperature problems.

Harmonic fcns solve heat problems

<u>Washers</u>: $\underbrace{\ln|z|}_{\ln r} = \mathrm{Re}\,\log z$ is <u>harmonic</u>



<u>Prob</u>: Solve heat prob. $u = \begin{cases} T_1 & \text{on } C_{R_0}(0) \\ T_2 & \text{on } C_{r_0}(0) \end{cases}$ (circles)

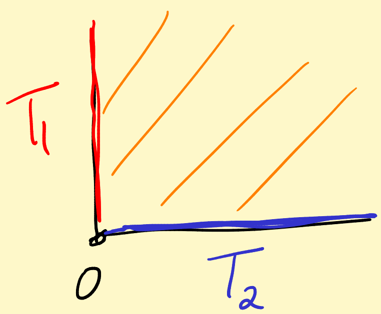
harmonic inside.

Sol^n $\quad \operatorname{Ln} r, \quad A \operatorname{Ln} r, \quad \underline{A \operatorname{Ln} r + B}$

works!

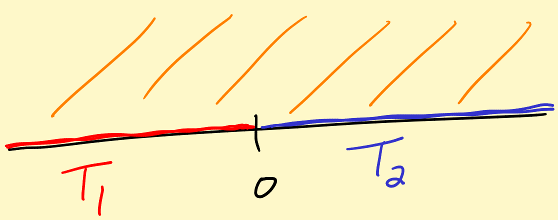
Need $\begin{cases} A \operatorname{Ln} r_0 + B = T_2 \\ A \operatorname{Ln} R_0 + B = T_1 \end{cases}$ get A, B!

Wedges, walls. $u = \operatorname{Arg} z = \operatorname{Im} \log z$ is harmonic!

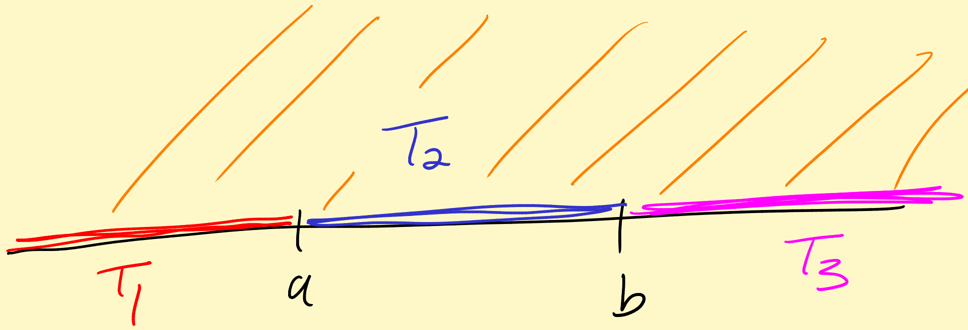


$u = \operatorname{Arg} z, \quad A \operatorname{Arg} z, \quad \underline{A \operatorname{Arg} z + B}$

works



Same plan works



Try $\quad A \operatorname{Arg}(z-a) + B \operatorname{Arg}(z-b) + C$