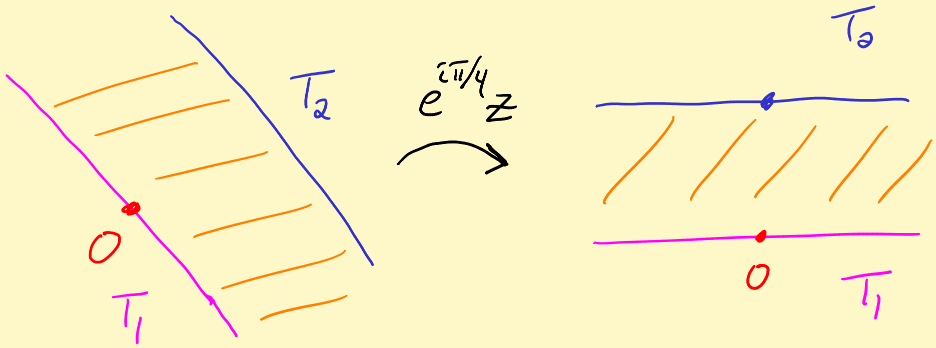
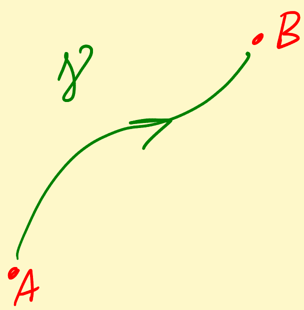
Lesson 11 Complex integration (Start Chap 4)

HWK hint:



Sol^n: $u(\underbrace{e^{i\pi/4}z}_{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)(x+iy)})$ $\qquad u = A \operatorname{Im} z + B$

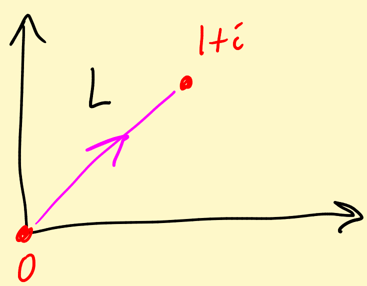
Curves in $\mathbb{C} \approx \mathbb{R}^2$



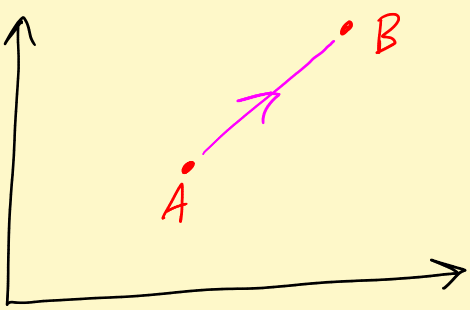
$\gamma: (x(t), y(t))$
$a \le t \le b$

In $\mathbb{C}$: $z(t) = x(t) + iy(t)$ $\qquad z: [a,b] \to \mathbb{C}$

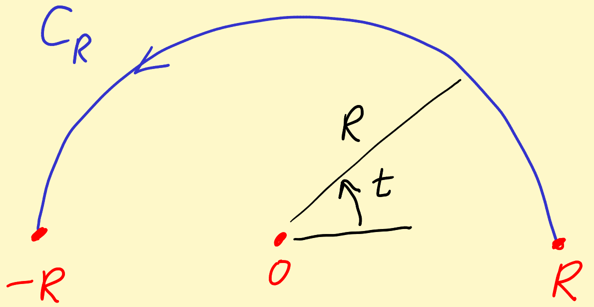
EX



$L: z(t) = t(1+i)$
$0 \le t \le 1$



$z(t) = A + t(B-A)$
$0 \le t \le 1$



$C_R: \ z(t) = Re^{it}$

$0 \leq t \leq \pi$

Two kinds of fcns 1) $f: \Omega \to \mathbb{C}$

analytic ($f = u + iv$, u, v C^1-smooth and satisfy C-R eqns $\Rightarrow$ DQ $\to f'$.)

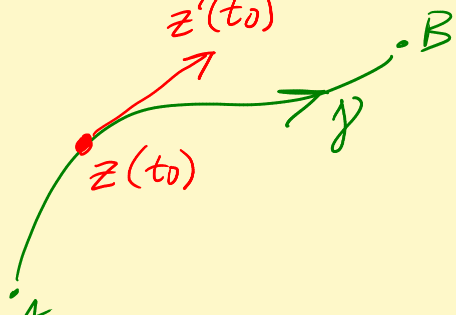
2) $z(t)$ $z: [a,b] \to \mathbb{C}$, $z(t) = x(t) + iy(t)$

Define $$z'(t_0) = \lim_{t \to t_0} \frac{z(t) - z(t_0)}{t - t_0}$$

$$= \lim_{t \to t_0} \left[\frac{x(t) - x(t_0)}{t - t_0} + i \frac{y(t) - y(t_0)}{t - t_0} \right]$$

$$= x'(t_0) + i y'(t_0)$$

Remark: $z'(t)$ represents the "velocity vector"



In pic. $z'(t_0)$ is a complex # that I put as if $z(t_0)$ is the origin.

Two kinds of integrals

1) Easiest kind $\int_a^b z(t)\,dt$

$$= \underbrace{\lim_{\Delta t \to 0} \sum_{n=1}^{N} z(t_n)\,\Delta t}_{\sum_{n=1}^{N} x(t_n)\Delta t \;+\; i \sum_{n=1}^{N} y(t_n)\Delta t}$$

$$= \int_a^b x(t)\,dt + i \int_a^b y(t)\,dt \quad \leftarrow \text{Def}^{n}.$$

Assume $x(t), y(t)$ cont.

... or piecewise cont.

Basic inequality #1 $\left| \int_a^b z(t)\,dt \right| \leq \left(\max_{a \leq t \leq b} |z(t)| \right)(b-a)$

Why: $\left| \int_a^b z(t)\,dt \right| = \left| \lim_{\Delta t \to 0} \sum_{1}^{N} z(t_n)\,\Delta t \right|$

$$\leq \underbrace{\lim_{\Delta t \to 0} \sum_{1}^{N} |z(t_n)|\,\Delta t}_{\int_a^b |z(t)|\,dt} \overset{\checkmark}{\leq} \left(\max |z(t)| \right)(b-a)$$

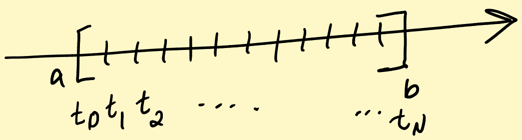
Baby lemma: $\int_a^b k\, z(t)\,dt = k \int_a^b z(t)\,dt$, $k \in \mathbb{C}$

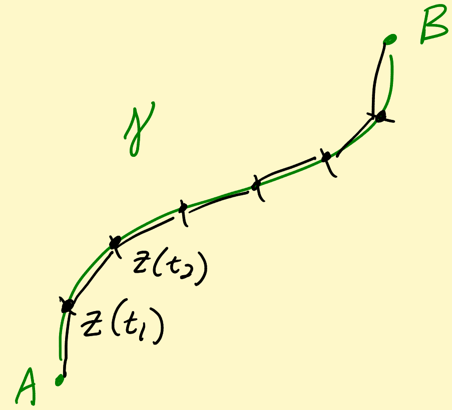
k const

Pf: $K = c_1 + i c_2$, $z(t) = x(t) + i y(t)$

Multiply, write out, both sides same. ✓

Integrate $f(z)$ along γ





$$\int_\gamma f\, dz = \lim_{\Delta t \to 0} \sum_{n=1}^{\infty} f(z(t_n)) \underbrace{\left(z(t_n) - z(t_{n-1})\right)}_{\Delta z}$$

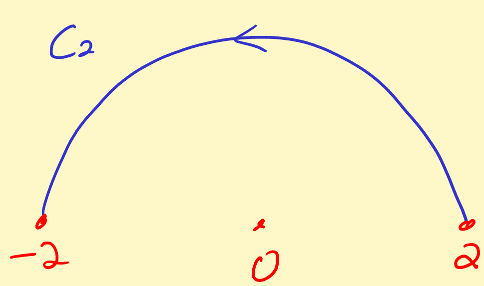
Hmmm. $$\Delta z = \underbrace{\frac{z(t_n) - z(t_{n-1})}{\Delta t}}_{z'(t_n)} \cdot \Delta t$$

$$\Delta z \approx z'(t)\, \Delta t$$

$$\int_\gamma f\, dz = \lim = \int_a^b f(z(t)) \underbrace{z'(t)\, dt}_{} \quad \leftarrow \text{Def}^{n}$$

Think $= dz = \frac{dz}{dt}\, dt$

EX



$C_2 : z(t) = 2e^{it}$

$0 \le t \le \pi$

$$z(t) = 2\cos t + i\,2\sin t$$

$$z'(t) = -2\sin t + i\,2\cos t = 2ie^{it}$$

$$\int_{C_2} e^z\,dz = \int_0^{\pi} e^{2e^{it}} \underbrace{\left(2ie^{it}\right)}_{z'(t)}\,dt$$

$$= \int_0^{\pi} e^{2\cos t + i2\sin t} \cdot 2ie^{it}\,dt$$

$$= 2i\int_0^{\pi} e^{2\cos t}\, \underbrace{e^{i2\sin t}\; e^{it}}_{e^{i(t+2\sin t)}}\,dt$$

$$= 2i\left[\int_0^{\pi} e^{2\cos t}\cos(t+2\sin t)\,dt + i\int_0^{\pi} e^{2\cos t}\sin(t+2\sin t)\,dt\right]$$

$$= \text{Ugh!}$$

Cool thing: Fund Thm Calc for analytic fcns =

$$\int_{C_2} e^z\,dz = \int_{C_2} \frac{d}{dz}\left(e^z\right)dz = e^z\Big|_{\text{START}}^{\text{END}}$$

$$= e^{-2} - e^{2}$$

Two chain rules 1) f, g analytic, then

$h(z) = f(g(z))$ is analytic and $h'(z) = f'(g(z))g'(z)$.

2) f analytic, $z(t) : [a,b] \to \mathbb{C}$

$$\frac{d}{dt} f(z(t)) = \underbrace{f'(z(t))}_{\text{complex deriv}} \underbrace{z'(t)}_{\text{"vector" deriv}}$$

Why $z(t) = x(t) + i y(t)$

$f(x+iy) = u(x,y) + i v(x,y)$

$$\frac{d}{dt} f(x(t) + i y(t)) = \frac{d}{dt}\left[u(x(t), y(t)) + i v(x(t), y(t))\right]$$

$$= \Big(u_x \dot{x} + \overset{\substack{-v_x \\ \shortparallel}}{u_y} \dot{y}\Big) + i \Big(v_x \dot{x} + \overset{\substack{u_x \\ \shortparallel}}{v_y} \dot{y}\Big) \qquad \text{C-R eqns}$$

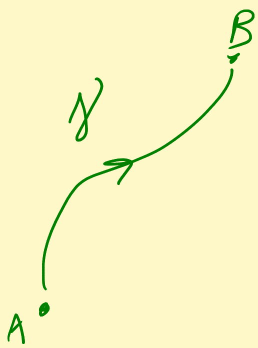
$$= \underbrace{\left[u_x + i v_x\right]}_{\text{red box \#1!}} \underbrace{(\dot{x} + i \dot{y})}_{z'(t)}$$

$$= f'(z(t)) \cdot z'(t) \quad \checkmark$$

Two Fundamental Theorems of Calculus

1) $$\underbrace{\int_a^b z'(t)\,dt}_{\text{easy}} = z(b) - z(a)$$

$$\text{easy} = \underbrace{\int_a^b x'(t)\,dt}_{x(b)-x(a)} + i \underbrace{\int_a^b y'(t)\,dt}_{y(b)-y(a)} = z(b) - z(a)$$

2) 

$$\int_\gamma f'(z)\,dz = f(B) - f(A)$$

when f analytic on an open set containing γ

Two basic estimates for $\int$'s

1) $$\left| \int_a^b z(t)\,dt \right| \le \left(\max_{a \le t \le b} |z(t)| \right)(b-a)$$

2) $$\left| \int_\gamma f\,dz \right| \le \left(\max_\gamma |f| \right) \text{Length}(\gamma)$$

Hmmm $\displaystyle\int_\gamma f'(z)\,dz = \int_a^b \underbrace{f'(z(t))\,z'(t)}_{\frac{d}{dt} f(z(t))}\,dt$

$$= f(z(t))\Big|_{t=a}^{b}$$

$$= f(B) - f(A) \quad \checkmark$$