Lesson 12 Complex integration

Office hour on Tues cancelled. Questions on Piazza.

Two kinds of functions: $w = f(z) \quad f: \Omega \to \mathbb{C}$

$$z(t) = x(t) + iy(t) \qquad z: [a,b] \to \mathbb{C}$$

Two kinds of derivatives: $f'(z)$ complex deriv

$$z'(t) = x'(t) + iy'(t)$$

Two kinds of integrals: $\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Two kinds of chain rules: $h(z) = f(g(z)). \quad h'(z) = f'(g(z))g'(z)$

$$\frac{d}{dt} f(z(t)) = f'(z(t))z'(t)$$

Two Fundamental theorems of Calculus:

1) $\int_a^b z'(t)\,dt = z(b) - z(a)$

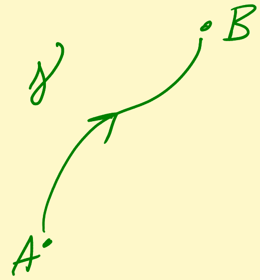
2) $\int_\gamma f'(z)\,dz = f(\text{END}) - f(\text{START})$

<u>Two Basic inequalities</u>: 1) $\left| \int_a^b z(t)\,dt \right| \le M \cdot (b-a)$

$M = \max_{t\in[a,b]} |z(t)|$

2) $\left| \int_\gamma f(z)\,dz \right| \le \left(\max_{z \in \gamma} |f(z)| \right) \text{Length}(\gamma)$

<u>Pf</u>: Fund thm calc



$$\int_\gamma f'(z)\,dz = f(B) - f(A)$$

Need f <u>analytic</u> on open Ω containing γ.

[Assuming u, v are C^1-smooth and satisfy C-R eqns.
Hence f' is <u>continuous</u> on γ.]

$$\int_\gamma f'(z)\,dz = \int_a^b f'(z(t))\,\underbrace{z'(t)\,dt}_{dz}$$

$$\underset{\substack{\text{Chain rule}\\ \#2}}{=} \int_a^b \frac{d}{dt}\Big[f(z(t))\Big]dt \underset{\substack{\uparrow\\ \text{Fund}\\ \text{Thm Calc}\\ \#1}}{=} f(z(b)) - f(z(a)) \quad \checkmark$$

<u>The Basic estimate for complex integrals</u>:

$$\left| \int_\gamma f(z)\,dz \right| \le M \cdot L$$

where $M = \max_\gamma |f| = \max_{t \in [a,b]} |f(z(t))|$ and

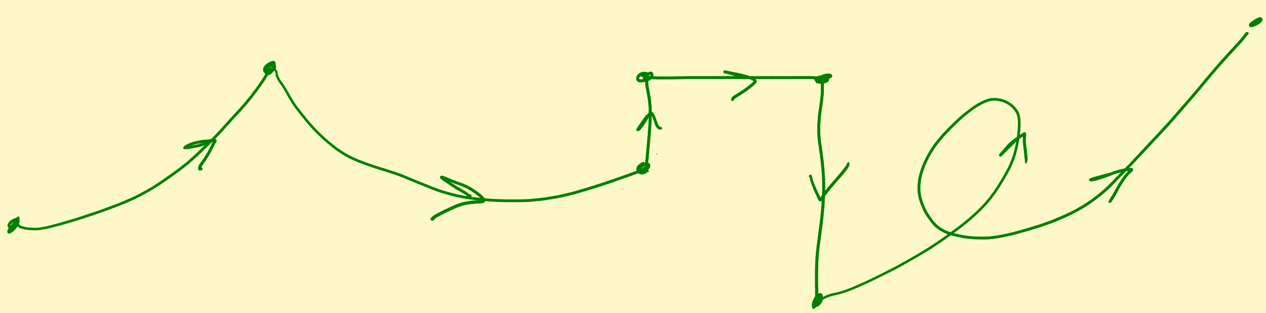
$$L = \text{Length}(\gamma) = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt$$

Pf $\left|\int_\gamma f\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$

$$\underset{\substack{\uparrow \\ \text{Pf of Basic est} \\ \#1}}{\leq} \int_a^b \underbrace{\left|f(z(t))\,z'(t)\right|}_{\leq M\cdot|z'(t)|}\,dt$$

$$\leq M \underbrace{\int_a^b \underbrace{|z'(t)|}_{\sqrt{(\dot{x})^2+(\dot{y})^2}}\,dt}_{=L}$$

Important fact I've been assuming γ is a C^1-curve : $x(t), y(t), \dot{x}(t), \dot{y}(t)$ exist and are continuous up to $[a,b]$. Easy to generalize our facts to piecewise C^1-smooth curves =



Continuous, but "corners" are allowed.

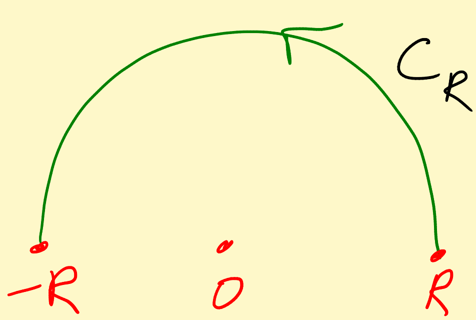
$$\int_\gamma f'\,dz = \sum_{1}^{N} \int_{\gamma_k} f'\,dz$$

$$= \left[f(z_2) - f(z_1)\right] + \left[f(z_3) - f(z_2)\right] +$$

$$+ \left[f(z_4) - f(z_3)\right] + \cdots + \left[f(z_N) - f(z_{N-1})\right]$$

$$= f(z_N) - f(z_1) \quad \checkmark$$

<u>EX</u>



$$C_R:\ z(t) = Re^{it}\,dt$$

$$z'(t) = ?$$

$$z(t) = R\,E(it)$$

$$z'(t) = R\,\underbrace{E'(it)}_{=E(it)}\,\frac{d}{dt}\left[it\right]$$

$$= Re^{it} \cdot i \quad \checkmark$$

$$\int_{C_R} e^z\,dz = \int_{C_R} \frac{d}{dz} e^z\,dz = e^z \Big|_R^{-R} = e^{-R} - e^{R}.$$

<u>Coming soon</u> Cauchy's theorem: f analytic on domain Ω with <u>no holes</u>, then $\int_\gamma f\,dz = 0$ for all <u>closed</u> curves γ in Ω.

Funny thing: "Reverse" is true: If $f : \Omega \to \mathbb{C}$ is continuous on Ω and $\int_\gamma f\,dz = 0$ for all closed curves γ in Ω, then f is analytic.

Must study $\int_\gamma f\,dz$!

Book: Read 4.4b (Green's thm approach to Cauchy's).
Skip 4.4a for now (homotopy approach)

Big fact: Suppose f is analytic on a domain Ω. Then f has a complex antiderivative F (such that $F'(z) = f(z)$ on Ω) if and only if

$$\int_\gamma f\,dz = 0 \text{ for every closed curve } \gamma \text{ in } \Omega.$$

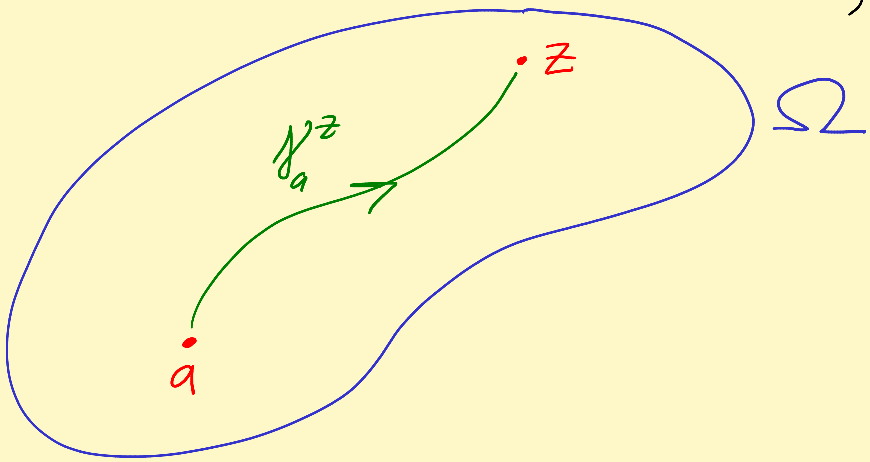
Pf ($\Rightarrow$) easy. Suppose $F'(z) = f(z)$.

$$\text{Then } \int_\gamma f\,dz = \int_\gamma F'(z)\,dz = F(\text{END}) - F(\text{START}) = 0$$

same
γ closed

($\Leftarrow$) We assume $\int_\gamma f\,dz = 0$, γ closed.

[Hmmm. If we had F with $F'(z) = f(z)$, we could write



$$\int_{\gamma_a^z} f(w)\,dw = \int_{\gamma_a^z} F'(w)\,dw = F(z) - F(a)$$

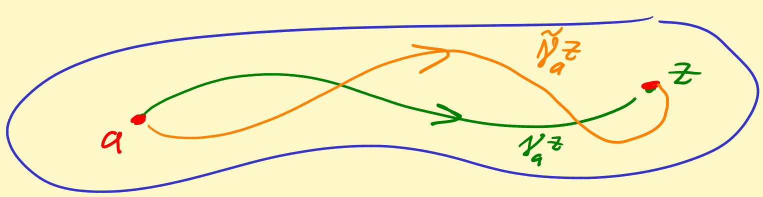
Aha! Think a fixed. z slides around.

$$F(z) = \int_{\gamma_a^z} f\,dw + \underbrace{F(a)}_{\text{const}}$$

$$\text{So} \quad \underbrace{F'(z)}_{f(z)} = \frac{d}{dz}\int_{\gamma_a^z} f\,dw + 0 \quad]$$

Big idea: "Define" $F(z) = \int_{\gamma_a^z} f(w)\,dw$

<u>Step 1</u> Show F is <u>well</u> defined.

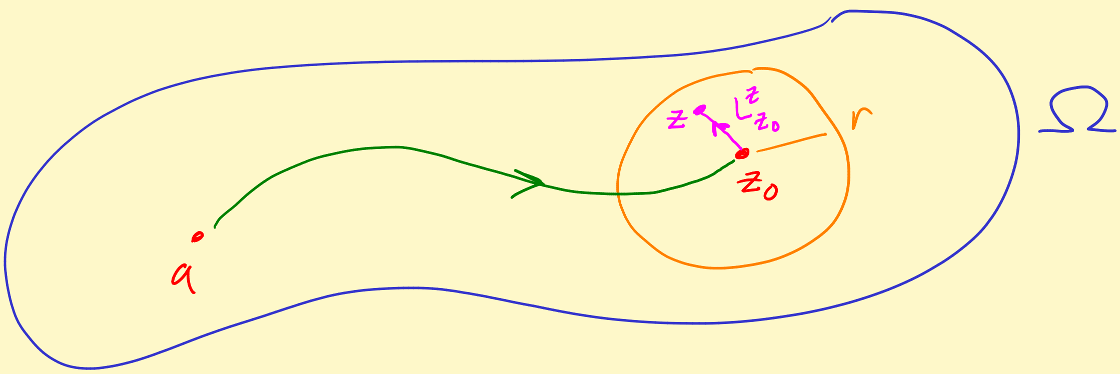


Aha! $\Gamma = (\gamma_a^z) \cup (-\tilde{\gamma}_a^z)$ is a <u>closed</u> curve

$$0 = \int_\Gamma f\,dw = \underbrace{\int_{\gamma_a^z} f\,dw - \int_{\tilde{\gamma}_a^z} f\,dw}_{\int\text{'s are } \underline{\text{same}}.}$$

<u>Step 2</u> Show that $F'(z) = f(z)$. Pick $z_0 \in \Omega$.

Ω open. So $\exists\ D_r(z_0) \subset \Omega$, $r > 0$.



$$F(z) = \underbrace{\int_{\gamma_a^{z_0}} f\,dw}_{\text{const}} + \underbrace{\int_{L_{z_0}^z} f\,dw}_{\text{Show } \frac{d}{dz} \text{ of this } = f(z)}$$

Next time