Lesson 13 The Cauchy theorem (4.4b)

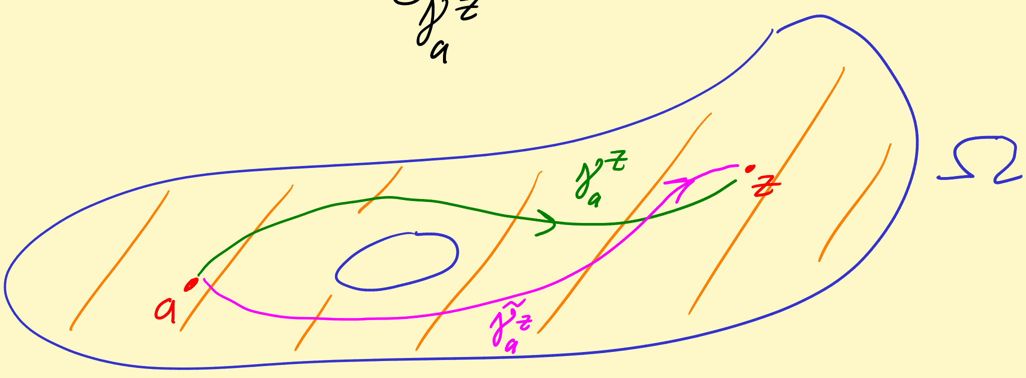
HWK 5 posted. Due next Wed.

Thm f analytic on a domain Ω. f has an analytic antiderivative on Ω (a fcn F such that $F'(z)=f(z)$) if and only if $\int_\gamma f\,dz = 0$ for all closed γ in Ω.

Pf: ($\Rightarrow$) easy. Fund thm calc. Last time.

($\Leftarrow$) Assume $\int_\gamma f\,dz = 0$ for closed γ in Ω.

"Define" $F(z) = \int_{\gamma_a^z} f(w)\,dw$



$\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ closed curve.

$$0 = \int_\Gamma f\,dw = \int_{\gamma_a^z} f\,dw + \underbrace{\int_{-\tilde{\gamma}_a^z} f\,dw}_{= -\int_{\tilde{\gamma}_a^z} f\,dz}$$

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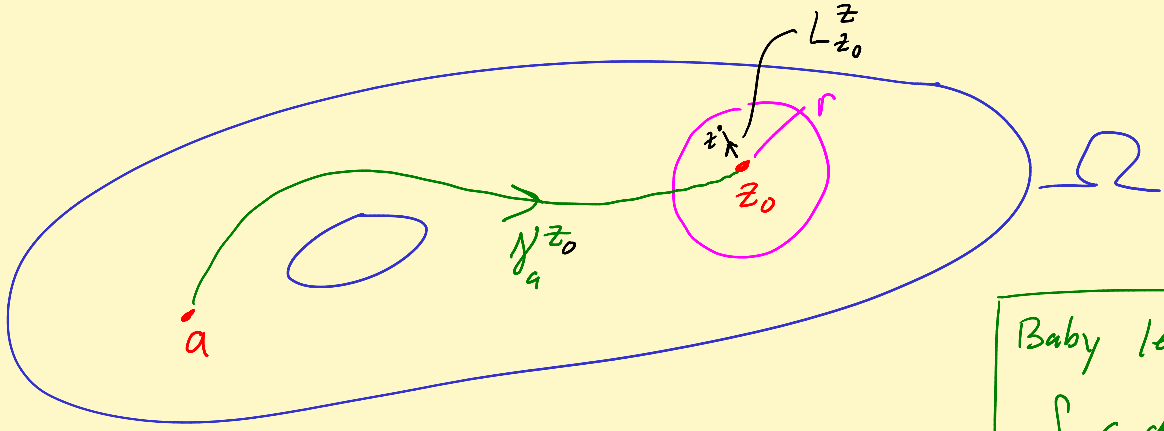
Well defined. ✓



Baby lemma:

$$\int_{L_{z_0}^{z}} c\,dw = \int_{L_{z_0}^{z}} \frac{d}{dw}[cw]\,dw$$

$$= [cw]_{z_0}^{z} = c(z-z_0)$$

$$F(z) = \int_{\gamma_a^{z_0}} f\,dw + \int_{L_{z_0}^{z}} f\,dw$$

$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_{L_{z_0}^{z}} \underbrace{f(w)}_{=f(z_0)+E(z)}\,dw$$

where $E(z) \to 0$ as $z \to z_0$

[because f' exists $\Rightarrow$ f continuous.]

$$= \frac{1}{z-z_0}\left[\underbrace{\int_{L_{z_0}^{z}} f(z_0)\,dw}_{f(z_0)(z-z_0)} + \int_{L_{z_0}^{z}} E(w)\,dw\right]$$

$$= f(z_0) + \underbrace{\frac{1}{z-z_0}\int_{L_{z_0}^{z}} E(w)\,dw}_{\mathcal{E}(z)}$$

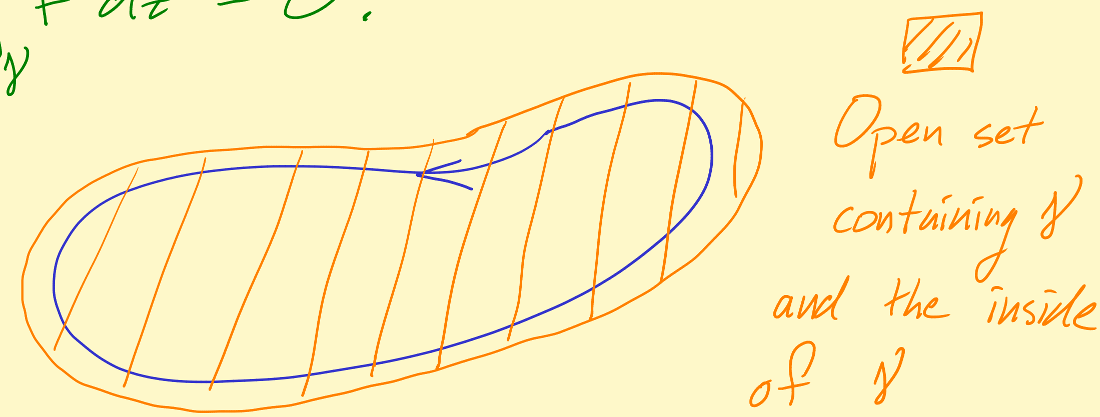
$$|\mathcal{E}(z)| = \frac{1}{|z-z_0|}\left|\int E\,dw\right| \le \frac{1}{|z-z_0|}\underbrace{\left(\max_{L_{z_0}^{z}}|E(w)|\right)}_{<\varepsilon}\underbrace{\mathrm{Length}(L_{z_0}^{z})}_{|z-z_0|}$$

Given $\varepsilon>0$, $\exists\ \delta>0$ such that $|E(w)|<\varepsilon$ when $|z-z_0|<\delta$.

So $\left|DQ - f(z_0)\right| < \varepsilon$ when $z \in D_\delta(z_0)$.

Cauchy's theorem Suppose f is analytic "inside and on" a simple closed curve γ.

Then $\int_\gamma f\,dz = 0$.



Green's theorem

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Pf of Cauchy's using Green's $\quad f(x+iy) = u(x,y) + iv(x,y)$

f analytic means u,v C^1-smooth and satisfy C-R eqns ($\Rightarrow f'(z)$ exists). Great! u,v C^1-smooth means Green's is good.

$\gamma : z(t) = x(t) + i\,y(t) \quad , \quad a \le t \le b$

$$\int_\gamma f\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$$= \int_a^b \Big[u(x(t),y(t)) + i\,v(x(t),y(t))\Big]\Big[\dot{x}(t) + i\dot{y}(t)\Big]\,dt$$

$$= \int_a^b (u\dot{x} - v\dot{y})\,dt + i\int_a^b (u\dot{y} + v\dot{x})\,dt$$

[Reminder: $\displaystyle\int_\gamma P\,dx + Q\,dy = \int_a^b P(x(t),y(t))\frac{dx}{dt}\,dt + Q(x(t),y(t))\frac{dy}{dt}\,dt$]

$$= \int_\gamma (\underset{P=u}{u}\,dx - \underset{Q=-v}{v}\,dy) + i\int_\gamma (\underset{Q=u}{u}\,dy + \underset{P=v}{v}\,dx)$$

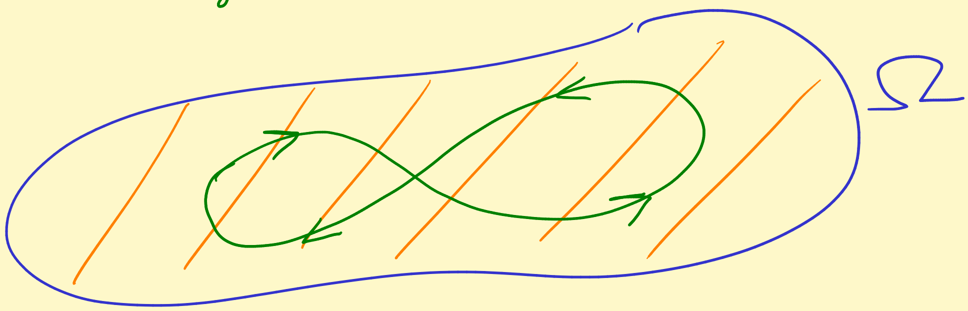
$$\underset{\text{Green's}}{=} \left(\iint_\Omega \underbrace{\left(\frac{\partial}{\partial x}(-v) - \frac{\partial u}{\partial y}\right)}_{\text{C-R eqn \#2}} dx\,dy + i\iint_\Omega \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\text{C-R Eqn \#1}} dx\,dy\right.$$

$$= 0$$

Cauchy thm #2 Suppose Ω is a domain with no holes (a "simply connected domain").

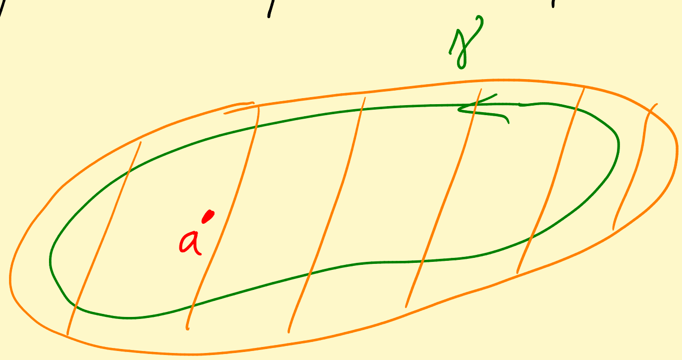
Given analytic f on Ω, and a closed curve γ in Ω, $\int_\gamma f\,dz = 0$.

Why:



Cor Previous thm about antiderivatives applies to show that on a domain with no holes, analytic fcns have analytic antiderivatives.

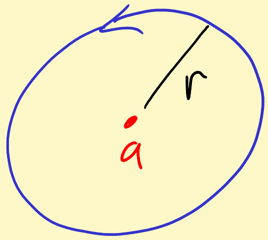
Exciting next step: Cauchy integral formula



f analytic on [shaded region]
γ simple closed curve
a inside γ

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$$

Important calculation



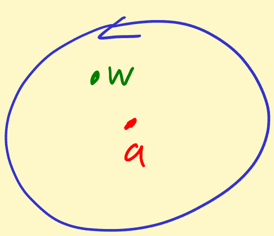
$C_r(a)$: $z(t) = a + re^{it}$
$0 \le t \le 2\pi$

$$\int_{C_r(a)} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{(a+re^{it})-a}\left[0+ire^{it}\right]dt$$

$$= \int_0^{2\pi} i\;dt = 2\pi i$$

<u>Note</u> Very important that f be analytic <u>inside</u> and on γ in Cauchy thm. $\frac{1}{z-a}$ is <u>not</u> analytic at a inside $C_r(a)$ and $\int_{C_r(a)} \frac{1}{z-a}\,dz = 2\pi i \neq 0$.

Hmmm. We also see that $\frac{1}{z-a}$ does <u>not</u> have an analytic antiderivative on $\Omega = \mathbb{C} - \{a\}$.



$C_r(a)$

Claim: $\int_{C_r(a)} \frac{1}{z-w}\,dz = 2\pi i$

for all w inside $C_r(a)$.

<u>Trick</u>: $H(w) = \int_{C_r(a)} \frac{1}{z-w}\,dz$

$$DQ = \int DQ \longrightarrow \int_{C_r(a)} \frac{1}{(z-w)^2}\,dz = H'(w)$$

Hmmm. $$H'(w) = \int_{C_r(a)} \frac{1}{(z-w)^2}\,dz = \int_{C_r(a)} \frac{d}{dz}\left[\frac{-1}{z-w}\right] dz$$

$$= \frac{-1}{z-w}\Bigg|_{z=\text{START}}^{\text{END}} = 0.$$

So $H' \equiv 0$ inside. It is constant! That constant is $2\pi i$ since $H(a) = 2\pi i$.