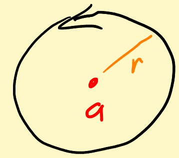
Lesson 14 The Cauchy integral formula

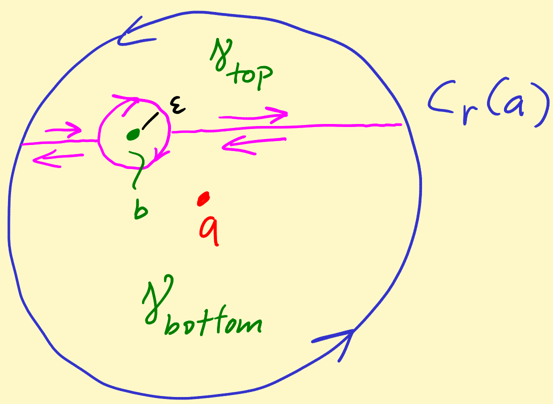
Last time: $C_r(a)$; $z(t) = a + re^{it}$, $0 \le t \le 2\pi$



$$\int_{C_r(a)} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{(a+re^{it})-a}\left[ire^{it}\right]dt = \int_0^{2\pi} i\,dt = 2\pi i$$

(*) Also, $\displaystyle\int_{C_r(a)} \frac{1}{z-b}\,dz = \begin{cases} 2\pi i & b \text{ inside } C_r(a) \\ 0 & b \text{ outside} \end{cases}$ ← by Cauchy thm

Another pf of (*)



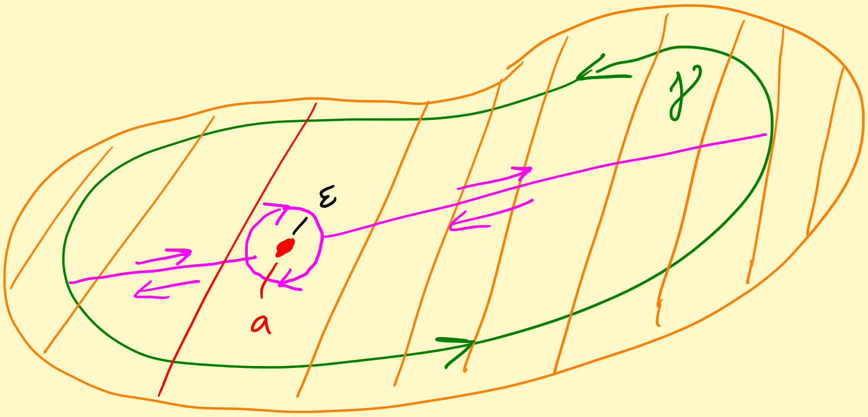
$$0 = \underbrace{\int_{\gamma_{top}} \frac{1}{z-b}\,dz}_{0 \text{ by Cauchy's}} + \underbrace{\int_{\gamma_{bottom}} \frac{1}{z-b}\,dz}_{=0 \text{ by Cauchy's}}$$

Add up all curve segments. Note that $\rightleftarrows$ cancel.

$$0 = \int_{C_r(a)} \frac{1}{z-b}\,dz + \underbrace{\int_{-C_\varepsilon(b)} \frac{1}{z-b}\,dz}_{= -2\pi i \text{ because clockwise.}}$$

Cauchy integral formula

γ simple closed curve, f analytic inside and on γ, a inside γ.



$$f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a}\,dz$$

Pf: Cauchy thm top, bottom:

$$0 = \underbrace{\int_{\gamma_{top}} \frac{f(z)}{z-a}\,dz}_{=0} + \underbrace{\int_{\gamma_{bottom}} \frac{f(z)}{z-a}\,dz}_{=0}$$

Cancel the $\rightleftarrows$

$$0 = \int_{\gamma} \frac{f(z)}{z-a}\,dz + \underbrace{\int_{-C_\varepsilon(a)} \frac{f(z)}{z-a}\,dz}_{= -\int_{C_\varepsilon(a)} \frac{f(z)}{z-a}\,dz}$$

Claim: Second term $\to 2\pi i\, f(a)$ ✓ as $\varepsilon \to 0$.

$$\left| \int_{C_\varepsilon(a)} \frac{f(z)}{z-a}\,dz - \underbrace{f(a)\,2\pi i}_{f(a)\int_{C_\varepsilon(a)} \frac{1}{z-a}\,dz \;=\; \int_{C_\varepsilon(a)} \frac{f(a)}{z-a}\,dz} \right|$$

$$= \left| \int_{C_\varepsilon(a)} \frac{f(z)-f(a)}{z-a}\, dz \right|$$

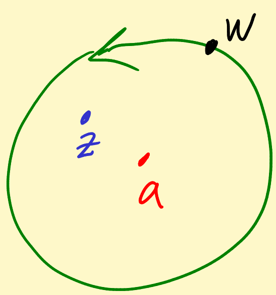
$$\leq \left(\underset{z\in C_\varepsilon(a)}{\mathrm{Max}} \frac{|f(z)-f(a)|}{\varepsilon} \right) \underbrace{\mathrm{Length}\,(C_\varepsilon(a))}_{2\pi\varepsilon}$$

$$\leq 2\pi \underset{C_\varepsilon(a)}{\mathrm{Max}} |f(z)-f(a)| \longrightarrow 0$$

because $f'(a)$ exists $\Rightarrow f$ continuous at a.

Harvest consequences right away!

On a disc: f analytic inside and on $C_r(a)$



$$f(z) = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f(w)}{w-z}\, dw$$

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{1}{2\pi i} \int_{C_r(a)} f(w) \left[\underbrace{\frac{\frac{1}{w-z} - \frac{1}{w-z_0}}{z-z_0}}_{\frac{\left(\frac{(w-z_0)-(w-z)}{(w-z)(w-z_0)}\right)}{z-z_0}} \right] dw$$

$$= \frac{1}{2\pi i} \int_{C_r(a)} f(w) \frac{1}{(w-z)(w-z_0)}\, dw$$

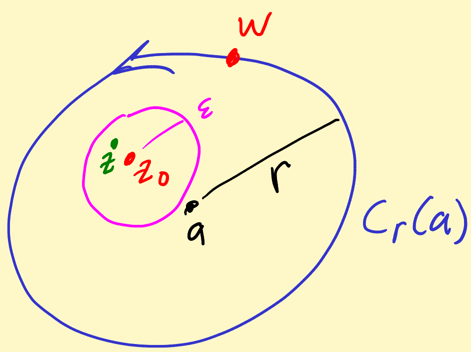
$$\xrightarrow{\text{Guess}} \quad \frac{1}{2\pi i}\int_{C_r(a)} \frac{f(w)}{(w-z_0)^2}\,dw$$

Aha! This is what you get when you diff under $\int$ in z-variable.

Show guess is correct:

$$\left|\frac{1}{2\pi i}\int_{C_r(a)} f(w)\left[\underbrace{\frac{1}{(w-z)(w-z_0)} - \frac{1}{(w-z_0)^2}}_{\dfrac{z-z_0}{(w-z)(w-z_0)^2}}\right]dw\right|$$

$$\leq \underbrace{|z-z_0|}_{\substack{\to 0 \\ \text{as } z\to z_0}}\ \frac{1}{2\pi}\underbrace{\left(\max_{C_r(a)}|f|\right)}_{\leq M}\underbrace{\left(\max_{w\in C_r(a)}\frac{1}{|(w-z)(w-z_0)^2|}\right)}_{\text{just need a bound for this. Easy!}} 2\pi r$$



Aha! $|z-w| \geq r-\varepsilon$

$|z_0-w| \geq r-\varepsilon$

Second Max $\leq \dfrac{1}{(r-\varepsilon)(r-\varepsilon)^2}$

Higher order Cauchy formulas:

$$f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\,dz$$

Mind blowing fact! Can keep doing this!

$$f''(a) = \frac{2}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^3}\, dz$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Promised big fact follows. f' exists $\Rightarrow f''$ exists $\Rightarrow f'''$ exists $\Rightarrow \cdots$

and $f^{(n+1)}$ exists $\Rightarrow f^{(n)}$ continuous.

<u>Consequently</u>: If $f(x+iy) = u(x,y) + iv(x,y)$ is <u>analytic</u> on open Ω, then u, v are C^∞-smooth on Ω, and u and v are <u>harmonic</u> fcns.

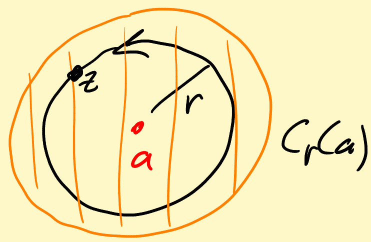
<u>Remark</u>: Taking $f(z) \equiv 1$ shows that result from last time $DQ_z \int \frac{1}{w-z}\, dw = \int DQ_z \frac{1}{w-z}\, dw$

$$\longrightarrow \int \frac{\partial}{\partial z}\left[\frac{1}{w-z}\right] dw = \int \frac{1}{(w-z)^2}\, dw$$

$$\underset{\text{Aha!}}{\overset{\uparrow}{=}} \int \frac{\partial}{\partial w}\left[-\frac{1}{w-z}\right] dw \underset{\substack{\text{Fund Thm} \\ \text{Calc}}}{\overset{\uparrow}{=}} 0$$

Next time: Liouville's theorem and the Fund Thm Algebra

Tool: Cauchy estimates:



$$\left|f^{(n)}(a)\right| = \left|\frac{n!}{2\pi i}\int_{C_r(a)} \frac{f(z)}{(z-a)^{n+1}}\,dz\right|$$

$$\leq \frac{n!}{2\pi}\left(\operatorname*{Max}_{C_r(a)} \frac{|f(z)|}{r^{n+1}}\right) 2\pi r$$

$$\leq \frac{n!\,M}{r^n} \quad \text{where } M = \operatorname*{Max}_{C_r(a)} |f|$$