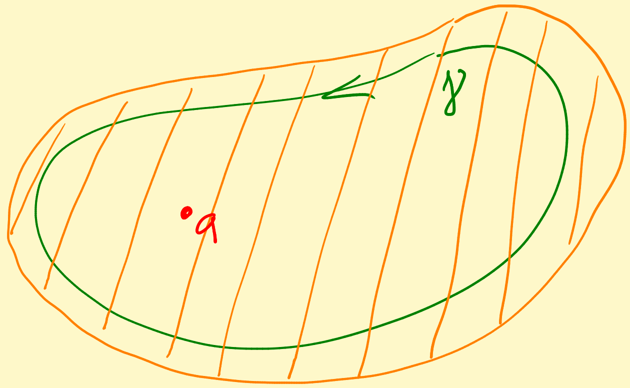
Lesson 15 Liouville's theorem & the Fundamental theorem of Algebra



f analytic on [shaded region]

a inside simple closed γ

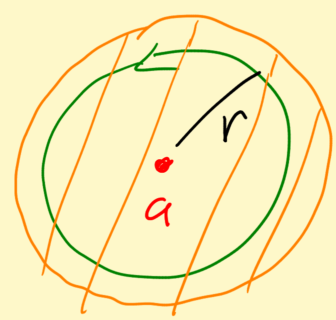
$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz \qquad f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{d}{da}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\,dz$$

$$\cdots \quad f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Thm: $\displaystyle \frac{d}{dz}\int_\gamma \frac{\varphi(w)}{w-z}\,dw = \int_\gamma \frac{\varphi(w)}{(w-z)^2}\,dw$ ← Diff under $\int$ in z-variable

γ curve segment or closed curve, φ continuous on γ.

Pf: Last time. ✓

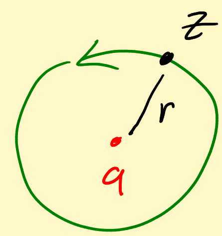


Cauchy estimates $\quad \gamma = C_r(a)$

$$\left|f^{(n)}(a)\right| \le \frac{n!\,M}{r^n} \quad \text{where} \quad M = \max_{C_r(a)} |f|$$

Way false $\mathbb{R} \to \mathbb{R}$. $\quad f(x) = \sin nx \quad |f(x)| \le 1$

Pf:
$$\left|f^{(n)}(a)\right| = \left| \frac{n!}{2\pi i} \int_{C_r(a)} \frac{f(z)}{(z-a)^{n+1}}\, dz \right|$$



$$\le \frac{n!}{2\pi} \left(\operatorname{Max}_{z \in C_r(a)} \frac{|f(z)|}{\underbrace{|z-a|^{n+1}}_{|z-a|=r}} \right) \underbrace{\text{Length}(C_r(a))}_{2\pi r}$$

$$\le \frac{n!\, M}{2\pi \; r^{n+1}}\, 2\pi r = \frac{n! M}{r^n} \checkmark$$

Liouville's theorem A bounded entire fcn must be constant.

Words: bounded means $|f(z)| \le M$ for all $z \in \mathbb{C}$.

entire means analytic on $\mathbb{C}$.

Way false $\mathbb{R} \to \mathbb{R}$. $\sin x$.

Pf: Cauchy estimate with $n=1$.

$$|f'(a)| \le \frac{1!\, M}{r^1}$$

Aha! Can let $r \nearrow \infty$.
$|f|$ bndd on $\mathbb{C}$ means same M works for all r!

Pick any a. $|f'(a)| \le \frac{M}{r}$. Let $r \nearrow \infty$. See

$f'(a)$ must $= 0$. a was arbitrary.

So $f'(a) \equiv 0$ on $\mathbb{C}$. So f is const. ✓

Next: Liouville's $\Rightarrow$ Fund thm alg.

Basic polynomial estimate $\quad P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$

where $a_N \neq 0$ and $N \geq 1$. $\leftarrow$ $P(z)$ not constant.

There are real constants $0 < A < B$ and a radius $R > 0$ such that

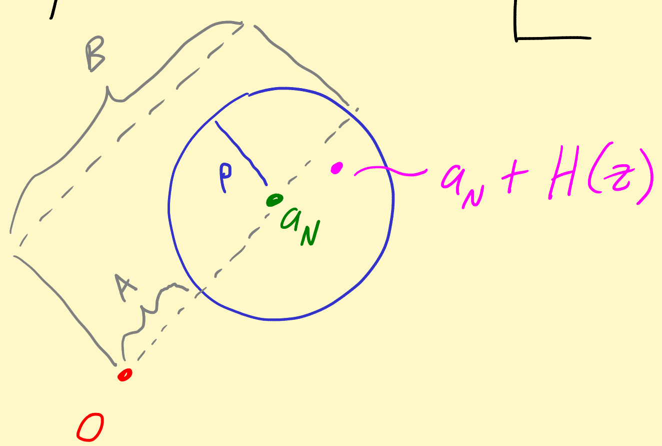
$$A|z|^N < |P(z)| < B|z|^N$$

when $|z| > R$.

Think: $|P(z)|$ blows up like $|z|^N$ as $z \to \infty$.

Note: All zeroes of $P(z)$ are outside $\overline{D_R(0)}$.

Why: $$P(z) = z^N \left[a_N + \underbrace{\frac{a_{N-1}}{z} + \frac{a_{N-2}}{z^2} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}}_{} \right]$$



$H(z) \to 0$ as $z \to \infty$.

Aha! Can choose a radius $R > 0$ so that $|H(z)| < \rho$ when $|z| > R$.

See $A < \left| \frac{P(z)}{z^N} \right| < B$. ✓

Just need $\rho < |a_N|$. Books: Why not take $\rho = \frac{|a_N|}{2}$.

Fun fact: Take very small ρ. See $0 < \underbrace{A < |a_N| < B}_{\text{close as desired.}}$

Fund thm of Alg A complex poly of degree $N \geq 1$ has a root in $\mathbb{C}$.

Why. Suppose $P(z)$ deg $N \geq 1$ does not have a root.

Then $\frac{1}{P(z)}$ entire! Basic poly est:

$$A|z|^N < |P(z)| < B|z|^N \text{ for } |z| > R.$$

So $\left| \underbrace{\frac{1}{P(z)}}_{f(z)} \right| < \underbrace{\frac{1}{A|z|^N}}$ if $|z| > R$

$< \frac{1}{AR^N}$ if $|z| > R$.

[Important fact A continuous real value fcn on a closed bounded set assumes its max value and is consequently bounded.]

$|f(z)|$ is continuous on $\overline{D_R(0)}$ ← closed, bndd

Bounded by some M.

So $|f|$ is bndd by $\mathrm{Max}(M, 1/AR^N)$ on $\mathbb{C}$.

f is bndd entire. Liouville's $\Rightarrow$ f const!

$P(z) = 1/f(z)$ is also const $\lightning$ contradiction!

[Why? $P(z)$ can't be const because $\left.\frac{d^N}{dz^N} P(z)\right|_{z=0} = N!\, a_N$ is <u>not</u> zero.]

So P must have a root.

<u>Cor</u>: Polys can be factored over $\mathbb{C}$.

Euclidean algorithm

$$\begin{array}{r|l} & q(z) \\ \hline P(z) & Q(z) \\ & \ \ \ddots \\ \hline & r(z) \end{array}$$

$r(z)$ ← remainder

$\underbrace{r(z)}_{\deg < \deg P}$

$$Q = Pq + r$$

$$\begin{array}{r|l} & q(z) \\ \hline (z-a) & Q(z) \\ & \ \ \ddots \\ \hline & b \end{array}$$

b ← const.

$$Q(z) = (z-a)\,q(z) + b$$

a root of Q: $\underbrace{Q(a)}_{0} = \underbrace{(a-a)\,q(a)}_{0} + b$

b must be 0.

So $Q(z) = (z-a)\,\underbrace{q(z)}$

$\deg q = \deg Q - 1$.

Repeat for q until you get $q(z) = Az + B$.

Conclude $P(z) = a_N \underbrace{(z-r_1)(z-r_2)(z-r_3)\cdots(z-r_N)}$

r's might repeat.

$$= a_N (z-r_1)^{m_1}(z-r_2)^{m_2}\cdots(z-r_\ell)^{m_\ell}$$

m_k = multiplicty of root r_k

$$\sum_{k=1}^{\ell} m_k = N$$