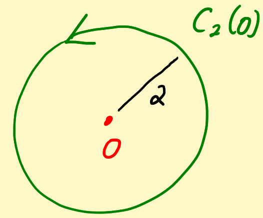
Lesson 16 Morera's theorem, Maximum principle

Fun problem $\quad C_2(0): z(t) = 2e^{it}, \quad 0 \le t \le 2\pi$



Compute $I = \displaystyle\int_{C_2(0)} \frac{e^{2z}}{(z-b)^3(z-a)}\, dz$

Case a, b outside $C_2(0)$: $I = 0$ by Cauchy's.

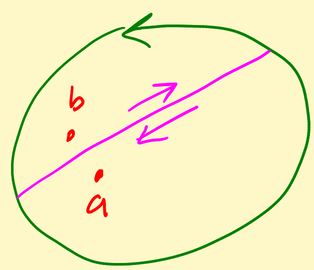
Case a inside, b outside: Cauchy integral formula with $f(z) = e^{2z}/(z-b)^3$. Get $I = 2\pi i\, f(a) = \dfrac{2\pi i\, e^{2a}}{(a-b)^3}$

Case b inside, a outside: Relevant formula:

$$f^{(2)}(b) = \frac{2!}{2\pi i} \int_{C_2(0)} \frac{f(z)}{(z-b)^{2+1}}\, dz. \qquad f(z) = \frac{e^{2z}}{(z-a)}$$

$$I = \frac{2\pi i}{2!} \frac{d^2}{dz^2} \left[\frac{e^{2z}}{z-a} \right] \Bigg|_{z=b}$$

Case a, b both inside.



Add up. Use previous cases.

or Partial fractions:

$$\frac{1}{(z-b)^3(z-a)} = \frac{A}{(z-b)^3} + \frac{B}{(z-b)^2} + \frac{C}{z-b} + \frac{D}{z-a}$$

Case $a=b$ inside. $\frac{2\pi i}{3!}\frac{d^3}{dz^3}\left[e^{2z}\right]\Big|_{z=a}$.

$(z-b)^4$ ← $n+1=4$, $n=3$

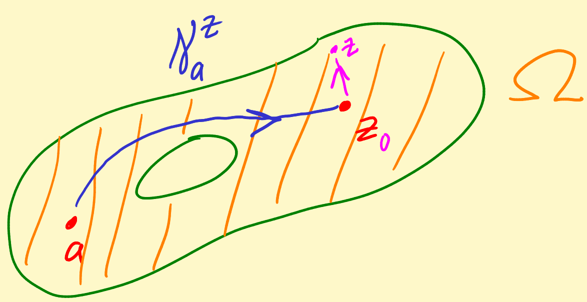
Know big fact: f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic ...

f cont. $\quad$ f' cont. $\quad$ ...

<u>Cor</u> (Morera's theorem) Given a <u>continuous</u> complex valued on a domain Ω. If $\int_\gamma f\,dz = 0$ for every closed γ in Ω, then f is analytic.

Remark: Reverse is true when Ω is <u>simply connected</u> (no holes).

<u>Pf of Morera's</u> Sneaky!



$$F(z) = \int_{\gamma_a^z} f(w)\,dw$$

Previously, showed F <u>well defined</u> if f analytic via the Cauchy thm. In this Cor, Morera's hyp gives us F well defined.

<u>Part 2</u> Works fine. Since f <u>continuous</u>, the

argument that showed $F'(z) = f(z)$ goes through

Part 3 Done. F' exists and $= f(z)$.

Big fact: So F'' exists too, i.e., f' exists. ✓

f is analytic.

Cool thing: We will use this to show that power series converge to analytic fcns:

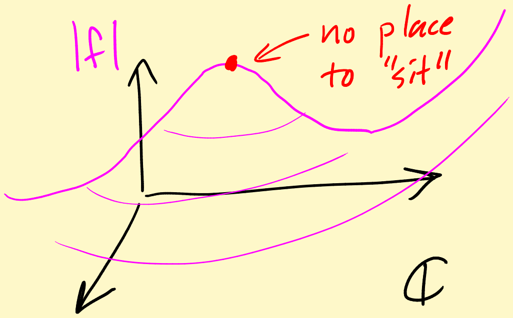
$$\int_\gamma \underbrace{\left[\sum_{n=0}^{\infty} a_n (z-z_0)^n\right]}_{\text{we will know p.s. converge uniformly.}} dz = \sum_{n=0}^{\infty} a_n \overbrace{\int_\gamma \underbrace{(z-z_0)^n}_{= \frac{d}{dz}\left[\frac{1}{n+1}\cdot(z-z_0)^{n+1}\right]} dz}^{=0} = 0$$

closed

Details later.

Maximum principle Suppose Ω is a bounded domain, f continuous on $\overline{\Omega}$, f analytic on Ω. Then $|f|$ attains its maximum value on the boundary of Ω.

Fancier version If $|f|$ has a local max at a pt inside Ω, then $f \equiv c$, a constant, on Ω.



Key Analytic fcns satisfy the averaging property.

f analytic on Ω, $\overline{D_r(a)} \subset \Omega$:

$$f(a) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta}_{\text{ave of } f \text{ on } \underbrace{C_r(a)}_{\text{circle}}}$$

$$= \frac{1}{\underbrace{2\pi r}_{\text{length } C_r(a)}} \int_0^{2\pi} f(a+re^{i\theta})\,\underbrace{r\,d\theta}_{ds} \leftarrow \text{arc length}$$

Why Cauchy int formula $\quad f(a) = \frac{1}{2\pi i}\int_{C_r(a)} \frac{f(z)}{z-a}\,dz$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{(a+re^{i\theta})-a}\left[ire^{i\theta}\right]d\theta$$

$$= \text{ave } \checkmark$$

Lemma If f is analytic on a disc and $|f|$ is constant there, then f is constant there.

Pf $\quad f = u + iv \qquad |f| = \sqrt{u^2+v^2} = c$, a constant.

better: $\quad |f|^2 = u^2 + v^2 = c^2 \quad (*)$

Hmmm: $\begin{cases} \frac{\partial}{\partial x}(*) : & 2u\,u_x + 2v\,v_x = 0 \\ \frac{\partial}{\partial y}(*) : & 2u\,u_y + 2v\,v_y = 0 \end{cases}$

$$\begin{bmatrix} u_x & v_x \\ \underset{\underset{-v_x}{\parallel}}{u_y} & \underset{\underset{u_x}{\parallel}}{v_y} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$$

Aha! $\quad \text{Det} = (u_x)^2 + (v_x)^2 = |\underbrace{u_x + i v_x}_{\text{red box \#1 for } f'}|^2$

Dumb case: $c = 0$. $u^2 + v^2 = 0 \Rightarrow u, v \equiv 0$. $f = 0$. ✓

Case $c \neq 0$. Then $\begin{pmatrix} u \\ v \end{pmatrix}$ is not the zero vector.

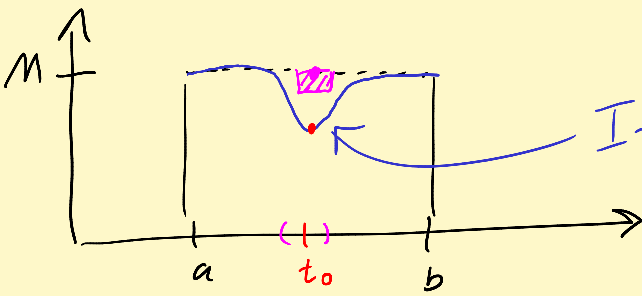
$\mathbb{A}\begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$. Must be that $\det \mathbb{A} = 0$.

But $\det \mathbb{A} = |f'|^2$. So $f' \equiv 0$ and we conclude that $f \equiv c$, const on disc.

Calculus lemma If $h(t)$ is continuous on $[a,b]$ and

$0 \le h(t) \le M$ there, then, if

$$\int_a^b h(t)\,dt = M(b-a), \quad \text{then } h(t) \equiv M \text{ on } [a,b].$$



If $h(t)$ dips below M, $\int_a^b h\,dt$ would be $< M(b-a)$.

Easier lemma. $h(t) \ge 0$, continuous.

$$\text{If } \int_a^b h(t)\,dt = 0, \quad \text{then } h(t) \equiv 0.$$