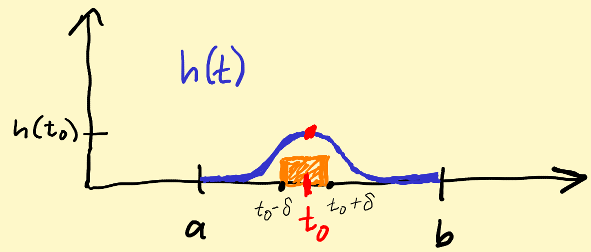
Lesson 17 Ave property and maximum principle for analytic & harmonic fcns.

<u>Easy calculus lemma:</u> $h : [a,b] \to \mathbb{R}$ cont, $h(t) \geq 0$ for all t.

If $\int_a^b h(t)\,dt = 0$, then $h(t) \equiv 0$.

<u>Pf</u>:



There is a $\delta > 0$ such that $|h(t) - h(t_0)| < \underbrace{\frac{h(t_0)}{2}}_{\varepsilon}$ when $|t - t_0| < \delta$.

Oops. $\int_a^b h(t)\,dt > 2\delta\, h(t_0)/2$. $\lightning$.

<u>Calculus lemma</u> $h : [a,b] \to \mathbb{R}$ cont and $0 \leq h(t) \leq M$.

If $\int_a^b h(t)\,dt = M(b-a)$, then $h(t) \equiv M$.

<u>Pf</u>: Apply easy lemma to $\tilde{h}(t) = M - h(t)$.

<u>Special case of max modulus princ.</u> Disc $D_r(a)$.

f analytic on $D_R(a)$ where $R > r$. Let $M = \operatorname{Max}\{|f(z)| : |z-a| \leq r\}$. If $|f(a)| = M$, then f is constant on $D_r(a)$.

<u>Pf</u>: $$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt$$

So $$\underbrace{|f(a)|}_{= M} \leq \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(a + re^{it})|}_{\leq M}\,dt \leq M$$

Aha! $2\pi M = \int_0^{2\pi} |f(a+re^{it})|\, dt = 2\pi M$

Calc lemma $\Rightarrow |f(a+re^{it})| \equiv M$.

Aha! Same holds for any radius ρ with $0<\rho<r$.

Conclude that $|f| \equiv M$ on $D_r(a)$.

Lemma: $|f|$ const $\Rightarrow f$ const on $D_r(a)$. ✓

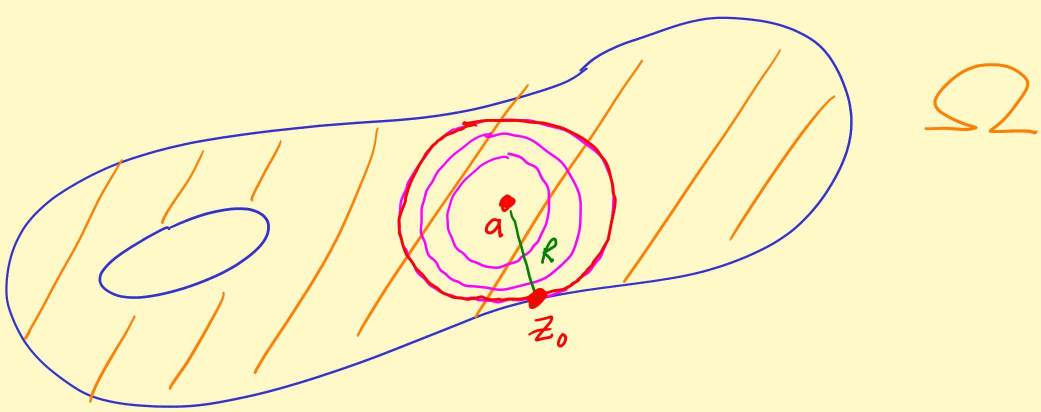
Max modulus thm Suppose Ω is a bounded domain and f is a continuous complex valued fcn on the closure $\overline{\Omega}$ of Ω which is analytic in Ω.

Then $M = \text{Max}\{|f(z)| : z \in \overline{\Omega}\}$ is assumed at some pt z_0 in bndry.

Note: $|f|$ continuous on closed and bounded $\overline{\Omega}$ $\Rightarrow$ $|f|$ attains max on $\overline{\Omega}$.

Pf: Case: M attained in bndry. Done. ✓

Case M attained at $a \in \Omega$



$M = |f(a)| = \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|\, dt = M$

by uniform continuity of $|f|$ on $\overline{D_R(a)}$, can let $r \nearrow R$.

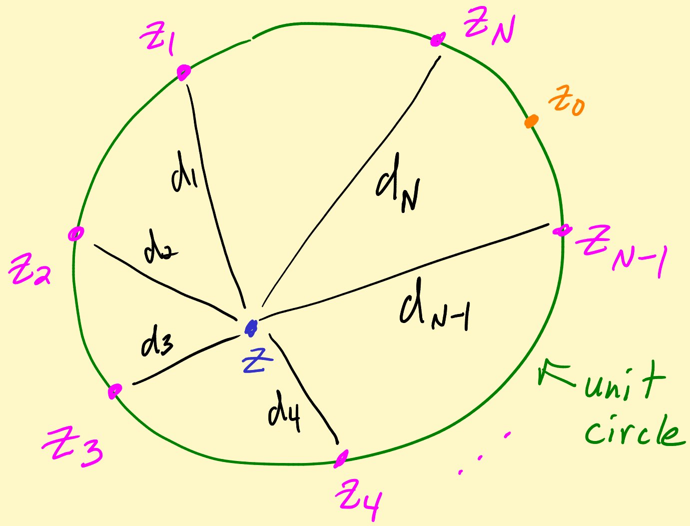
So get pt $z_0 \in$ bndry where $|f(z_0)| = M$. ✓

Fancier version. If Ω is a domain, f analytic on Ω, and $|f|$ has a local max at a, then $f \equiv$ const. on Ω.

Pf: Disc version shows there is a (possibly small) disc $D_r(a) \subset \Omega$ where $f \equiv$ const on $D_r(a)$.

Fact from power series: f const on $D_r(a) \subset \Omega \Rightarrow f$ const on Ω. We will do this later.

Problem



$$D(z) = \prod_{n=1}^{N} d_n$$

There is z on the unit circle where $D(z) = 1$.

Big idea: $f(z) = \prod_{n=1}^{N} (z - z_n)$ poly of deg N.

$$|f(z)| = \prod_{n=1}^{N} \underbrace{|z - z_n|}_{d_n} = D(z).$$

$f(z) = z^N + \cdots$. (So f is not constant.)

Aha! $|f(0)| = \prod_{n=1}^{N} |z_n| = 1.$

Hmmm. So $\max_{\overline{D_1(0)}} |f| = M \geq 1.$

Know max attained at some z_0 with $|z_0| = 1$.

If $M = 1$. Done. $|D(z_0)| = 1$.

If $M > 1$. Hmmm. $D(z_n) = 0$. $D(z_0) > 1$.

Aha! $D(e^{it})$ is a continuous fcn of $t \in [0, 2\pi]$.

Intermediate value thm $\Rightarrow$ there is a θ where $D(e^{i\theta}) = 1$.

Minimum modulus princ? No! z^2 has a min modulus of 0 at 0 on $\overline{D_1(0)}$. However, if f analytic and non-vanishing, a local min of $|f|$ inside $\Omega \Rightarrow f \equiv$ const. [Apply max princ to $1/f$.]

Fact: Averaging properties $\Rightarrow$ Max princ.

Harmonic fcns satisfy ave prop!

Suppose u harmonic ($\mathcal{C}^2$-smooth, $\Delta u \equiv 0$) on $D_R(a)$.

Know how to cook up harmonic v such that $f = u + iv$ is analytic on $D_R(a)$. Let $0 < r < R$

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{f(a+re^{it})}_{u+iv}\, dt$$

$$\underbrace{u(a)}+iv(a) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})dt} + i\frac{1}{2\pi}\int_0^{2\pi} v(a+re^{it})dt$$

$=$

✓

Can do straightforward repeat of analytic version.
Since no modulus involved, get Max u is attained on bndry.
Apply to $-u$ to see that min is also attained on bndry.

Cool trick to avoid repeating argument.

$$f = u+iv$$

Then $e^f = e^{u+iv} = e^u e^{iv}$

$$|e^f| = \underbrace{|e^u|}_{e^u}\underbrace{|e^{iv}|}_{1} = e^u \qquad (*)$$

u has a local max $\Rightarrow |e^f|$ has a local max $\Rightarrow$
e^f const. Hmmm. Does that imply $u \equiv$ const?
Yes! $(*)$ shows $e^u = |e^f|$. So $u = \text{Ln}|e^f|$. ✓