Lesson 18 More about harmonic functions (Practice problems posted)

Useful fact: If u is C^2-smooth and $\Delta u \equiv 0$ on a domain Ω, then

$$f(x+iy) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) \text{ is analytic on } \Omega.$$

$$f = \underbrace{u_x}_{U} - i\, u_y \qquad V = -u_y$$

C-R eqns: 1) $\frac{\partial U}{\partial x} \overset{?}{=} \frac{\partial V}{\partial y}$

$u_{xx} \overset{?}{=} -u_{yy}$ Yes! $\Delta u \equiv 0$.

2) $\frac{\partial U}{\partial y} \overset{?}{=} -\frac{\partial V}{\partial x}$

$\frac{\partial^2 u}{\partial y \partial x} \overset{?}{=} +\frac{\partial^2 u}{\partial x \partial y}$ Yes! u C^2

Great application If u is a C^2-smooth (real valued) harmonic on a domain Ω with no holes (Ω is a simply connected domain). Then u has a harmonic conjugate on Ω.

Hmmm. $f = u + iv$

$$f' = \begin{cases} u_x + i v_x & \#1 \\ v_y - i u_y & \#2 \end{cases} = \underline{\underline{u_x - i u_y}}$$

! Know this is analytic!

Big idea: Also know we can cook up analytic antiderivatives on domains with no holes.

Find an antiderivative f for $F = u_x - i u_y$.

$$f(z) = \int_{\gamma_a^z} F(w)\,dw \qquad \text{works.}$$

$$f'(z) = F(z).$$

Write $f = \bar{U} + i\bar{V}$

$$f' = \begin{cases} \bar{U}_x + i\bar{V}_x \\ \bar{V}_y - i\bar{U}_y \end{cases} = u_x - iu_y$$

See $\nabla \bar{U} = \nabla u$. So $u - \bar{U} = c$, a const.

C-R eqns!
$$\begin{cases} u_x = \bar{V}_y \\ u_y = -\bar{V}_x \end{cases}$$

So $u + i\bar{V}$ is analytic on Ω.

$\bar{V}$ is a harm conj for u.

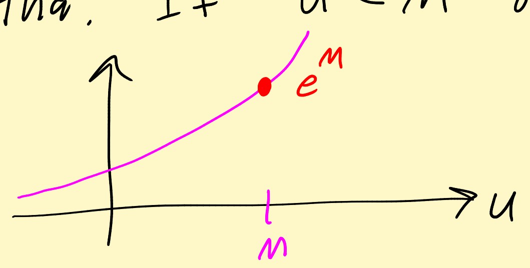
Liouville's theorem for harmonic fcns Suppose u is harmonic on $\mathbb{C}$ and bounded there. Then u must constant.

Pf: Get a harmonic conj v for u on $\mathbb{C}$.

$f = u + iv$ is analytic. Trick: $e^f = e^{u+iv} = e^u e^{iv}$

$$|e^f| = \underbrace{|e^u|}_{e^u}\ \underbrace{|e^{iv}|}_{1} = e^u$$

Aha! If $u < M$ on $\mathbb{C}$, then $|e^f| < e^M$



(e^u is strictly increasing)

e^f is entire and bndd $|e^f| < e^M$ on $\mathbb{C}$.

Liouville's $\Rightarrow$ $e^f \equiv c$, a constant.

Hmmm $e^u = |e^f| \equiv |c|$. So $u \equiv \ln|c|$. ✓

Question Is there a "Cauchy integral" formula for harmonic fcns? Yes! Poisson integral formula.

(Short expository paper about Poisson integral on home page.)

Suppose u is harmonic on $D_R(0)$ with $R>1$.

Get analytic $f = u + iv$ on $D_R(0)$. $a \in D_1(0)$.

$C_1(0)$: $z(t) = 0 + 1 \cdot e^{it}$, $0 \le t \le 2\pi$.

Know $$f(a) = \frac{1}{2\pi i} \int_{C_1(0)} \frac{1}{z-a} f(z)\, dz$$

Want to take real part, but it's a mess.

Trick: $$0 \overset{\text{by Cauchy thm}}{=} \frac{1}{2\pi i} \int_{C_1(0)} \frac{\bar{a}}{\underbrace{1 - \bar{a} z}} f(z)\, dz$$

set $=0$: $z = 1/\bar{a}$ outside $C_1(0)$!

$\dfrac{\bar{a}}{1-\bar{a}z} f(z)$ is analytic inside and on unit circle

Miracle:
$$f(a) = \frac{1}{2\pi i} \int_{C_1(0)} \Big[\frac{1}{z-a} + \underbrace{\frac{\bar{a}}{1-\bar{a}z}}_{\text{adding zero here}} \Big] f(z)\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \left(\frac{1}{e^{it}-a} + \frac{\bar{a}}{1-\bar{a}e^{it}} \right) f(e^{it})\, ie^{it}\, dt$$

$$\vdots$$

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|a|^2}{\underbrace{|e^{it}-a|^2}_{\text{real!}}} f(e^{it})\, dt$$

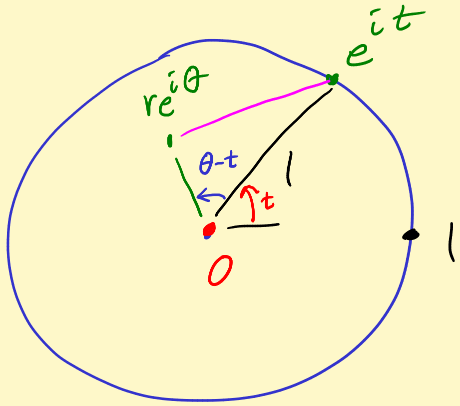
We can now take the real and imag parts!

Poisson formula →
$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|a|^2}{|e^{it}-a|^2} u(e^{it})\, dt$$

The Poisson formula solves the <u>Dirichlet problem</u>:

Given a continuous real valued fcn φ on $[0, 2\pi]$ with $\varphi(0) = \varphi(2\pi)$, there is a real valued <u>continuous</u> fcn u on $\overline{D_1(0)}$ such that $u(e^{i\theta}) = \varphi(\theta)$ for $0 \le \theta \le 2\pi$, and u is <u>harmonic</u> on $D_1(0)$.

Other forms:



/ line $= |re^{i\theta} - e^{it}|$

$$= \sqrt{1 + r^2 - 2r\cos(t-\theta)}$$

Law of cosines

/ line $= r$

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\, u(e^{it})\,dt$$

Do it on a circle of radius R about z_0.

$$u(z_0 + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(t-\theta)}\, u(z_0 + Re^{it})\,dt$$