Lesson 19 Power series

Great news: $\mathbb{R} \to \mathbb{R}$ proofs about limits, sequences, series go over to $\mathbb{C} \to \mathbb{C}$ line by line.

Power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

Ex Geometric series

$$\begin{aligned} 1 + z + z^2 + \cdots + z^N &= S_N \\ \times z \qquad z + z^2 + \cdots + z^N + z^{N+1} &= z S_N \\ \hline 1 - z^{N+1} &= (1-z) S_N \end{aligned}$$

$$S_N(z) = \frac{1}{1-z} - \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

Denominator estimate: $|1 - z| \geq \big|1 - |z|\big|$

Case: $|z| < 1$. Then $\big|1 - |z|\big| = 1 - |z|$

$$|E_N(z)| = \left| \frac{z^{N+}}{1-z} \right| = \frac{|z|^{N+1}}{|1-z|} \leq \frac{|z|^{N+1}}{1-|z|}$$

$\longrightarrow 0$ as $N \to \infty$ when $|z| < 1$.

Case $|z| \geq 1$: Then $|z^N| = |z|^N \geq 1$.

Oops! The terms of a convergent series __must__ go to zero as $N \to \infty$. Geom series does not conv for $|z| \geq 1$.

__Geom series__: "Radius of convergence" is $R = 1$.

__Important fact__ Geom series conv fast and __uniformly__ when $|z| < r < 1$.

In this case $|E_N(z)| \leq \frac{|z|^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-r}$

$$\boxed{\begin{array}{l} |z| < r \\ -|z| > -r \\ 1-|z| > 1-r \\ \frac{1}{1-|z|} < \frac{1}{1-r} \end{array}}$$

Aha! $\underbrace{\left| S_N - \frac{1}{1-z} \right|}_{} = \left| -E_N(z) \right| < \frac{r^{N+1}}{1-r} \to 0$ as $N \to \infty$

(Indep of z with $|z| < r$) <

Thm: $\sum_{n=1}^{\infty} z_n$ converges $\Rightarrow z_n \to 0$ as $n \to \infty$.

[Reverse not true. $\sum_1^{\infty} \frac{1}{n}$ diverges even though $\frac{1}{n} \to 0$]

$$S_N = \sum_1^N z_n$$

$$S_N - S_{N-1} = z_N$$

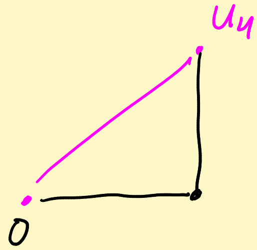
$$L - L = 0$$

<u>Def^n</u> $\sum_{n=1}^{\infty} u_n$ <u>converges absolutely</u> if $\sum_{n=1}^{\infty} |u_n| < \infty$.

<u>Thm</u> Absolute convergence $\Rightarrow$ convergence.

[In $\mathbb{R}$, this a consequence of the <u>completeness</u> of $\mathbb{R}$: every Cauchy seq converges. ($\mathbb{C}$ is complete too.)

<u>Pf</u>: Write $u_n = x_n + i y_n$. $|\text{Re } u_n| \le |u_n|$, $|\text{Im } u_n| \le |u_n|$



So $\left.\begin{array}{l} |x_n| \le |u_n| \\ |y_n| \le |u_n| \end{array}\right\}$ convergence of $\sum_1^{\infty} |u_n|$

$\Rightarrow \sum |x_n|, \sum |y_n| < \infty$ too.

(by comparison test MA 341)

Know $\sum x_n$ and $\sum y_n$ converge, say to X and Y.

Know $\sum_1^N u_n = \sum_1^N x_n + i \sum_1^N y_n \to X + iY$.

<u>Ratio test</u> $\sum_{n=0}^{\infty} u_n$. Suppose $\left|\frac{u_{n+1}}{u_n}\right| \to L$ as $n \to \infty$.

$$\begin{cases} L < 1 & \text{series converges \underline{absolutely}} \\ L > 1 & \text{diverges.} \\ L = 1 & \text{test fails.} \end{cases}$$

Freshman calc pf works. $L<1$. Compare tail end of series: $(0 \le L < r < 1)$ to geom series $\sum r^n$.

$L>1$: terms don't go to zero.

EX What is the radius of conv of $\sum_{n=0}^{\infty} \underbrace{n 3^n z^n}_{u_n}$?

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)3^{n+1} z^{n+1}}{n\, 3^n\, z^n}\right| = \frac{n+1}{n}\cdot 3 \cdot |z|$$

$$\longrightarrow 3|z| \begin{cases} <1 & \text{conv} \\ >1 & \text{div} \end{cases}$$

Convergence: $|z| < \frac{1}{3}$

Div: $|z| > \frac{1}{3}$.

R. of C. $= \frac{1}{3}$

EX $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \underbrace{\frac{z^n}{n!}}_{u_n} + \cdots$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}\right| = \frac{n!}{(n+1)!}\cdot|z| = \frac{1}{n+1}|z| \to 0 < 1$$

Ratio test $\Longrightarrow$ series converges for all z.

We say $R = \infty$.

EX: $\sum_0^\infty (n!)^n z^n$. Only converges if $z=0$!

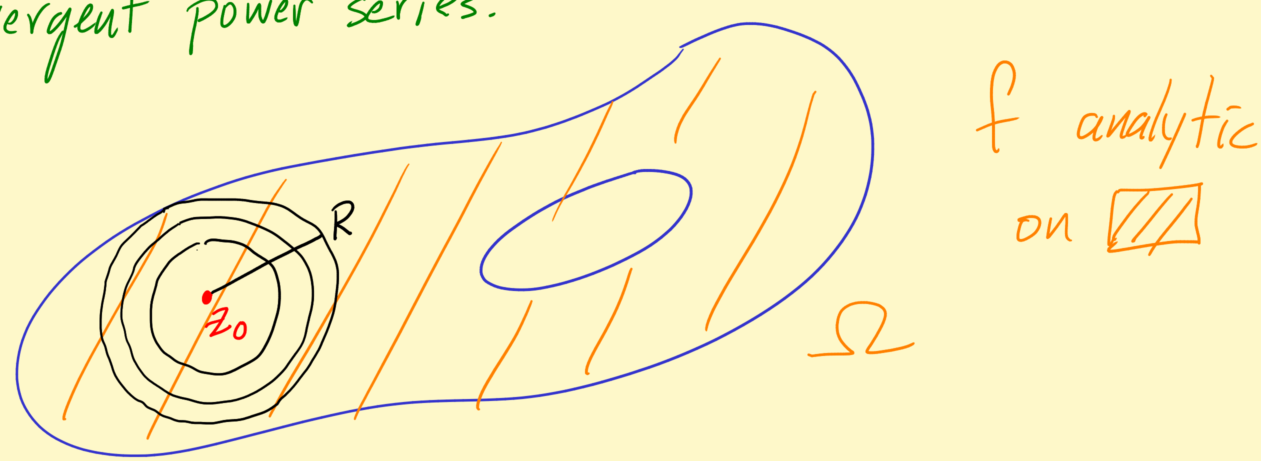
Say $R=0$.

Careful: Ratio test doesn't always work.

EX: Let $\{a_n\}_{n=1}^\infty$ be an ennumeration of complex #'s in $D_1(0)$ with rational real and imaginary parts.

$\sum_{n=1}^\infty a_n z^n$ has R. of C. $= 1$. Ratio test bombs.

Exciting fact Cauchy integral formula shows that analytic functions are given (locally) by absolutely convergent power series:

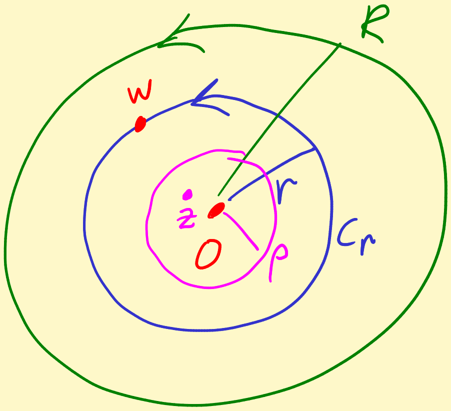


$D_R(z_0)$ = "biggest disc about z_0 in Ω"

Then $f(z) = \sum_{n=0}^\infty a_n (z-z_0)^n$ where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

and the R. of C. is $\geq R$.

Plan



$$f(z) = \frac{1}{2\pi i} \int_{C_r} f(w) \underbrace{\left[\frac{1}{w - z} \right]}_{} \, dw$$

$$\frac{1}{w} \cdot \frac{1}{\left(1 - \frac{z}{w}\right)}$$

$$\frac{1}{w}\left(1 + \left(\frac{z}{w}\right) + \left(\frac{z}{w}\right)^2 + \left(\frac{z}{w}\right)^3 + \cdots \right)$$