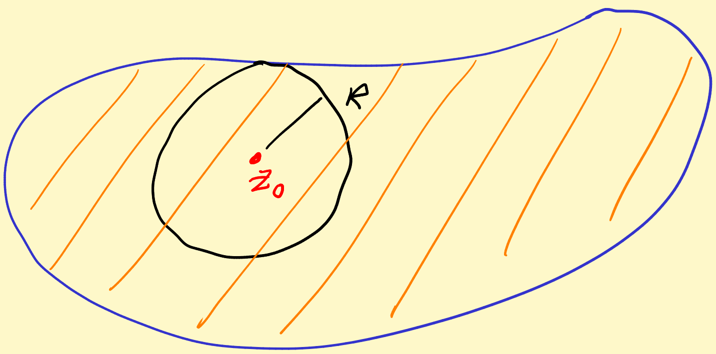
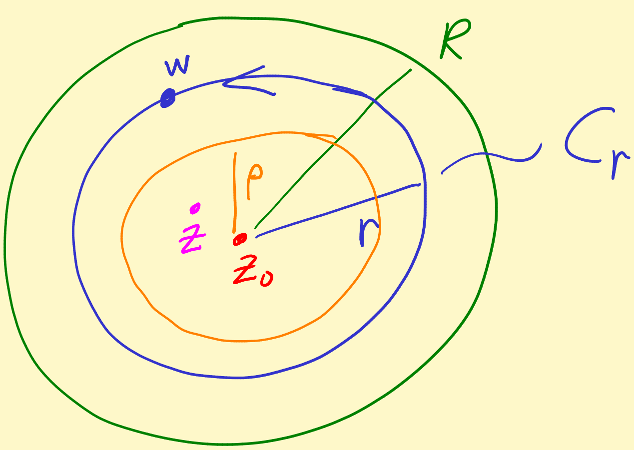
Lesson 20 Power series

$$\underbrace{1 + z + z^2 + \cdots + z^N}_{S_N(z)} = \frac{1 - z^{N+1}}{1-z} = \frac{1}{1-z} - \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

$$\boxed{\frac{1}{1-z} = S_N(z) + E_N(z)}$$



f analytic on Ω



Let $z_0 = 0$.

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w - z}\, dw$$

$$= \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w} \frac{1}{1 - \frac{z}{w}}\, dw$$

$$= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w}\left(\left[1+\left(\frac{z}{w}\right)+\left(\frac{z}{w}\right)^2+\cdots+\left(\frac{z}{w}\right)^N\right]+E_N\left(\frac{z}{w}\right)\right)dw$$

$$= \sum_{n=0}^{N}\underbrace{\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{\underset{\uparrow\,(w-0)^{n+1}}{w^{n+1}}}\,dw\right)}_{\text{Higher order Cauchy}} z^n + \underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}E_N\left(\frac{z}{w}\right)dw}_{\mathcal{E}_N(z)}$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_{C_r}\frac{f(w)}{(w-a)^{n+1}}\,dw$$

Wow! z^n coeff is $\frac{f^n(0)}{n!}$, Taylor's formula!

Note: $\left|\frac{z}{w}\right| = \frac{|z|}{r} < \underline{\underline{\frac{\rho}{r} < 1}}$

So $$\left|\mathcal{E}_N(z)\right| = \left|\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}E_N\left(\frac{z}{w}\right)dw\right|$$

$$\leq \frac{1}{2\pi}\max_{|w|=r}\frac{|f(w)|}{r}\cdot\left(\frac{\left(\frac{\rho}{r}\right)^{N+1}}{1-\left(\frac{\rho}{r}\right)}\right)2\pi r$$

$$\leq M\frac{\left(\frac{\rho}{r}\right)^{N+1}}{1-\left(\frac{\rho}{r}\right)} \quad \text{where } M = \max_{C_r}|f|.$$

$\longrightarrow 0$ as $N \to \infty$ uniformly in z on $D_\rho(0)$.

Def^n A seq of fcns $f_n(z)$ on a domain Ω is said to "converge uniformly on compact subsets of Ω" to f, if:

Given $\varepsilon > 0$ and compact $K \subset \Omega$ (meaning K is closed and bounded in Ω), there is an $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \varepsilon \quad \text{for } n > N \text{ and}$$

all $z \in K$. ["uniform" in $z \in K$.]

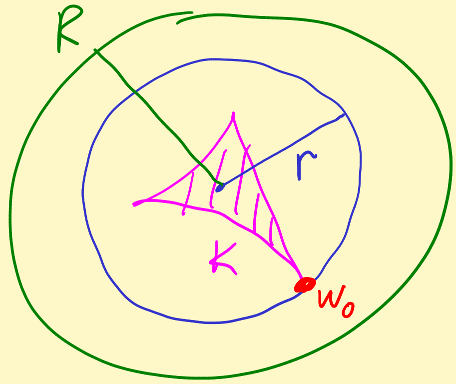
Write $f_n \twoheadrightarrow f$.

Remark When $\Omega = D_R(z_0)$, $f_n \twoheadrightarrow f$ means that $f_n \to f$ uniformy on each $D_r(z_0)$, $0 < r < R$.

Why: $\overline{D_r(z_0)}$ is a compact subset of $D_R(z_0)$.

Easy dir: $(\Longrightarrow)$

($\Leftarrow$). Let K be a compact subset of $D_R(z_0)$.



Hmmm. $z-z_0$ is continuous. Hence, so is $|z-z_0|$. A continuous fcn on K attains its max r.

$$r = \max_{w \in K} |w - z_0| = |w_0 - z_0| < R$$

See unif conv on K from unif conv on $D_{r+\varepsilon}(z_0)$ where $0 < \varepsilon < R - r$.

Fact from MA 341 A uniform limit of continuous fcns is continuous.

Fantastic fact If f_n are analytic on a domain Ω and $f_n \rightrightarrows f$, then f is analytic!

Pf: We know analytic fcns are continuous. So f is continuous. Restrict attention to a disc $D_R(z_0) \subset \Omega$.

Fantastic trick. Morera's thm! Suppose

γ is a closed curve in $D_R(z_0)$.

$tr(\gamma) =$ "Trace of γ" $= \{ z(t) : \gamma : z(t),\ a \le t \le b \}$

is a <u>closed</u> and <u>bounded</u> subset of $D_R(z_0)$.

$$\left| \int_\gamma f\, dz \right| = \left| \int_\gamma f\, dz - \underbrace{\int_\gamma f_n\, dz}_{\substack{=0 \text{ by} \\ \text{Cauchy thm}}} \right|$$

$$= \left| \int_\gamma f - f_n\, dz \right|$$

$$\le \underbrace{\left(\operatorname{Max}_\gamma |f - f_n| \right)}_{\substack{\to 0 \text{ as } n \to \infty \\ \text{because } f_n \to f \text{ \underline{unif}} \\ \text{on } tr(\gamma).}} \operatorname{Length}(\gamma)$$

So $\int_\gamma f\, dz = 0$ and Morera's $\Longrightarrow$ f analytic on $D_R(z_0)$ ✓

Hmmm. Know Analytic $\Longrightarrow$ (= power series locally)

Want reverse. Need to know that when power series converge, they do so <u>uniformly</u> on subdiscs. They do!

<u>Famous fact</u>: A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has a <u>radius of convergence</u> R, $0 \le R \le +\infty$, given by Hadamard's formula:

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

with properties $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

1) Converges <u>absolutely</u> in $D_R(z_0)$

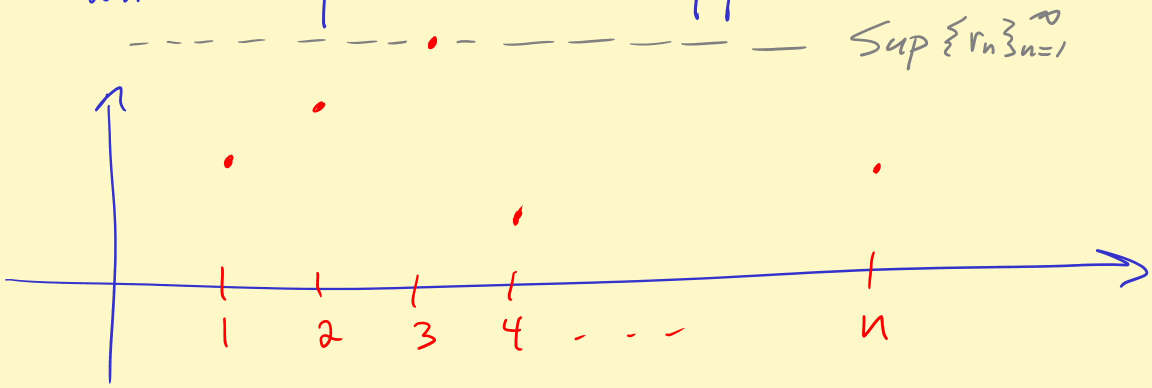
2) converges <u>uniformly</u> on $D_r(z_0)$ when $0<r<R$,

3) diverges when $|z-z_0|>R$.

<u>Def^n</u> $(r_n)_{n=1}^{\infty}$ is a seq of real #'s $r_n \ge 0$.

$$\limsup_{n\to\infty} = \lim_{N\to\infty} \sup\{r_n : n \ge N\}.$$

where $\sup A =$ "least upper bound" for A.



$\underbrace{\sup\{r_n : n \geq N\}}$ can only go <u>down</u> or stay same as N grows.

non-increasing seq bounded <u>below</u> by 0.

So it has a limit $= \text{Lim Sup}$,