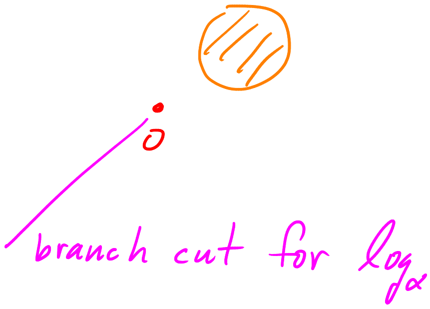
Review for Exam 1 in-class Wed., Feb. 28.

See home page for exam instructions about crib sheet etc.

3. Find a continuous real valued function u on the annulus $\{z : 1 \le |z| \le 2\}$ that is harmonic inside the annulus, equal to 20 on the inner boundary and equal to 5 on the outer boundary.

$$\operatorname{Ln}|z| = \operatorname{Re}\left[\log_\alpha z\right]$$



is locally the real part of an analytic, so harmonic on $\mathbb{C} - \{0\}$.

Try $\quad T(z) = A\operatorname{Ln}|z| + B$

want $\stackrel{?}{=} 20$

Need $\quad T(z) = A\operatorname{Ln}1 + B = B$ on $|z| = 1$

$T(z) = A\operatorname{Ln}2 + B = 5$ on $|z| = 2$.

$$A = \frac{5-20}{\operatorname{Ln}2} \qquad B = 20.$$

4. Use the Cauchy-Riemann equations to prove that a real valued analtyic function on a domain must be constant.

$f(x+iy) = u(x,y) + i\underbrace{v(x,y)}$ analytic

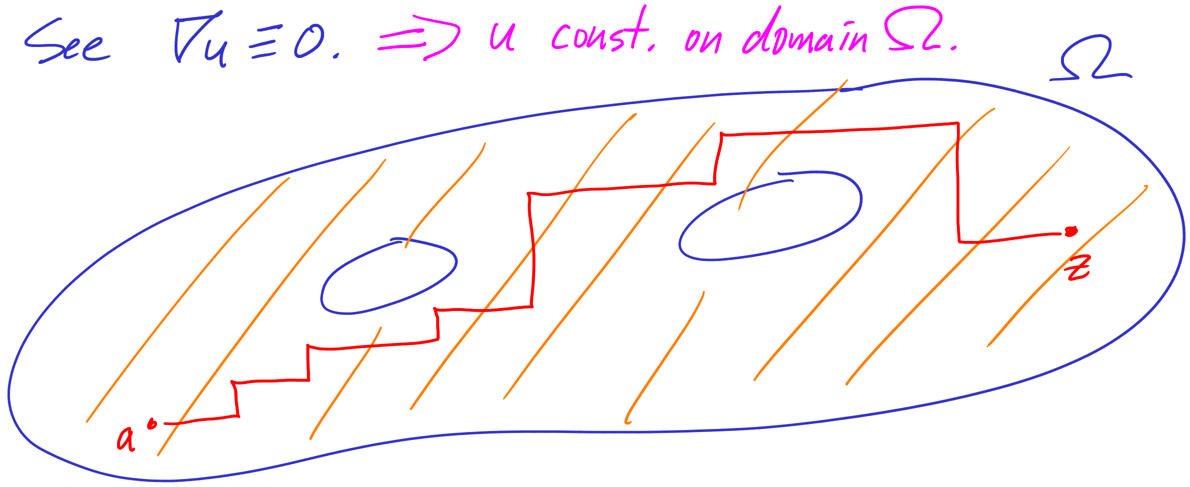
f real valued $\Rightarrow$ $\boxed{v \equiv 0}$.

Know u, v satisfy C-R equs:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \equiv 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \equiv 0$$

See $\nabla u \equiv 0$. $\Rightarrow$ u const. on domain Ω.



$|f(z)| = c$ $\qquad f = u + iv$

$\boxed{u^2 + v^2 = c^2}$ $(*)$ $\qquad \begin{cases} \frac{\partial}{\partial x}(*) \\ \frac{\partial}{\partial y}(*) \end{cases}$ Use C-R equs

$$\underbrace{\begin{bmatrix} u_x & v_x \\ -v_x & u_x \end{bmatrix}}\begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$$

Det $= u_x^2 + v_x^2 = |f'|^2$ must $\equiv 0$. $\quad$ not zero vector if $c \neq 0$.

6. Let C denote the unit circle parameterized in the counterclockwise sense. Compute $\displaystyle\int_C \frac{e^{3z}}{(2z-1)^2(z-2)}\,dz$. Explain.

$$= \frac{e^{3z}}{(z-2)} \cdot \frac{1}{2^2(z-\frac{1}{2})^2} \qquad 2 = n+1$$

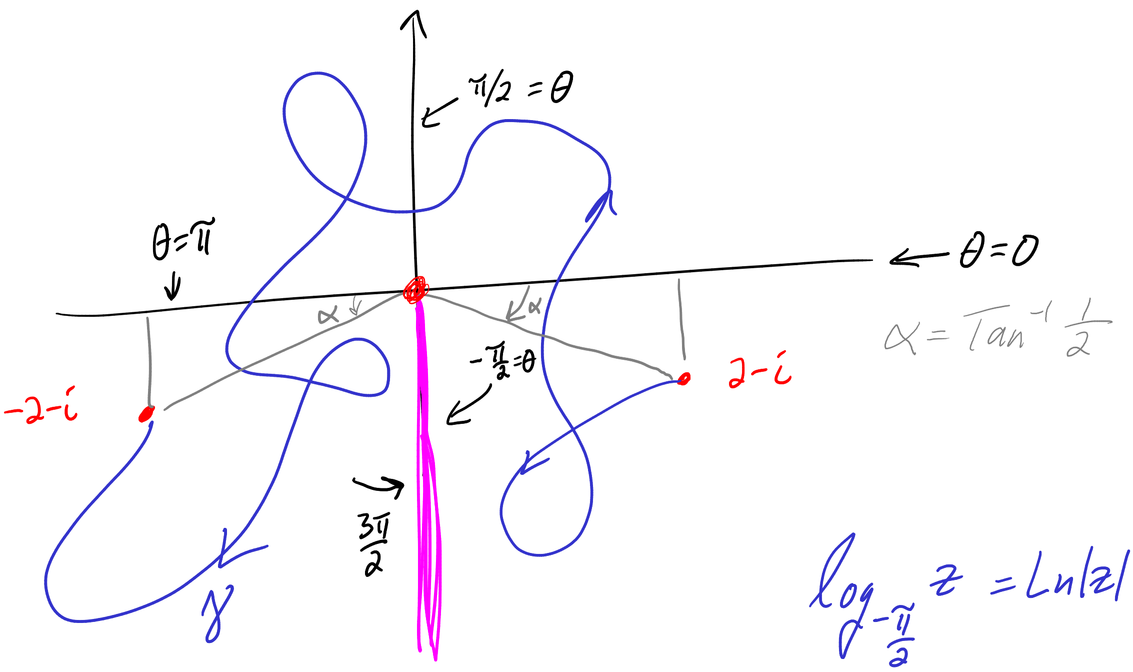
$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$f'(\tfrac{1}{2}) = \frac{1!}{2\pi i}\int_C \frac{\left[\frac{e^{3z}}{4(z-2)}\right]}{(z-\frac{1}{2})^2}\,dz$$

f(z)

Ans: $(2\pi i)\,1!\,f'(\frac{1}{2})$.

11. Let γ denote any curve that starts at $2-i$ and ends at $-2-i$ and avoids the set $\{it : t \leq 0\}$. Compute a) $\displaystyle\int_\gamma \frac{1}{z^3}\,dz$ b) $\displaystyle\int_\gamma \frac{1}{z}\,dz$



$$\log_{-\frac{\pi}{2}} z = \text{Ln}|z| + i\theta$$

$$\theta \in \arg z,\ -\frac{\pi}{2} < \theta < \underbrace{-\frac{\pi}{2} + 2\pi}_{\frac{3\pi}{2}}$$

$$\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}\left[\log_{-\frac{\pi}{2}} z\right] dz$$

$$= \log_{-\frac{\pi}{2}}(-2-i) - \log_{-\frac{\pi}{2}}(2-i)$$

$$= \left[\text{Ln}\sqrt{5} + i(\pi+\alpha)\right] - \left[\text{Ln}\sqrt{5} - i\alpha\right]$$

$$= i\left(\pi + 2\,\text{ArcTan}\,\frac{1}{2}\right)$$

$$\int_\gamma \frac{1}{z^3}\,dz = \int_\gamma \frac{d}{dz}\left[\frac{z^{-3+1}}{-3+1}\right] dz = \left[-\frac{1}{2} z^{-2}\right]_{\text{START}}^{\text{END}}$$

12. Let C_R denote the half circle parameterized by $z(t) = Re^{it}$ for $0 \le t \le \pi$. Show that

$$\left| \int_{C_R} \frac{1}{z^4+1}\, dz \right| \le \left(\max_{C_R} \frac{1}{|z^4+1|} \right)(\pi R)$$

tends to zero as R tends to infinity.

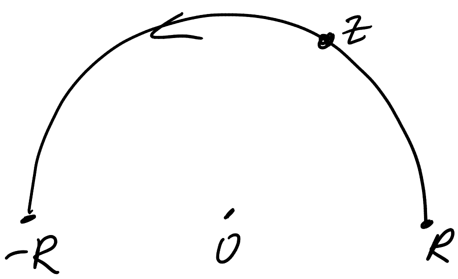
Basic estimate for complex $\int$'s:

$$\left| \int_\gamma f\, dz \right| \le \left(\max_\gamma |f| \right) \text{Length}(\gamma)$$

Numerator estimate
Denominator est

$$\left| \frac{a+b}{d+c} \right| \le \frac{|a|+|b|}{\big||d|-|c|\big|}$$

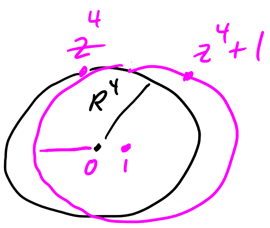
$$|d+c| = |d-(-c)| \ge \underbrace{\big||d|-|-c|\big|}_{\big||d|-|c|\big|}$$



$$\left| \frac{1}{z^4+1} \right| \le \frac{1}{\underbrace{\big||z^4|-1\big|}_{|R^4-1| = R^4-1 \text{ when } R>1.}}$$

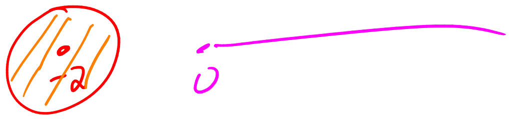
So $|I| \le \frac{1}{R^4-1} \cdot \pi R$ when $R>1$.

$\longrightarrow 0$ as $R \to \infty$.



13. Determine a branch of $\log(z^2 + 4z + 1)$ that is analytic near $z = -1$ and find its derivative there.

$1-4+1=-2 \leftarrow \mathrm{Log}$, Princ branch out.



$\log_0 z = \mathrm{Ln}|z| + i\theta$

$\theta \in \arg z,\ 0<\theta<2\pi$

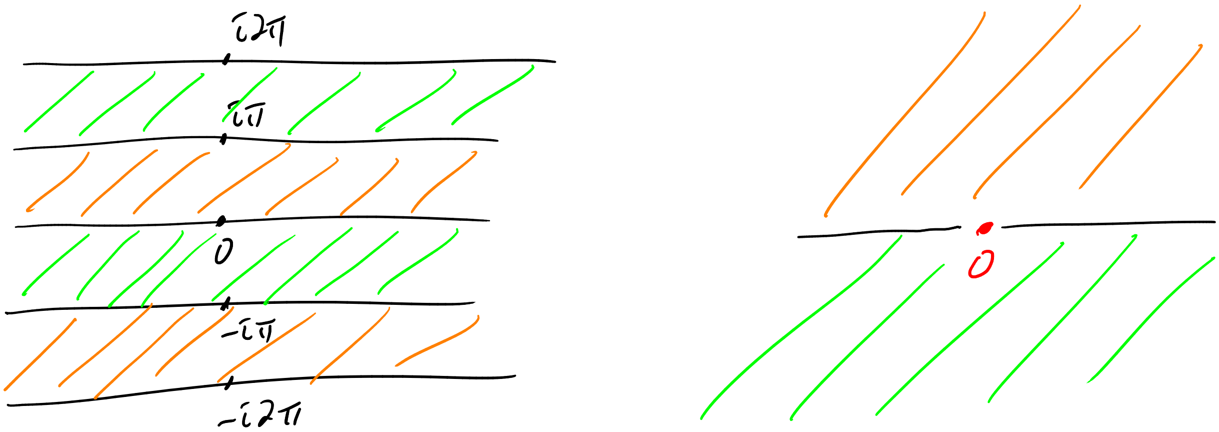
$\log_0(z^2+4z+1)$ is analytic near $z=-1$.

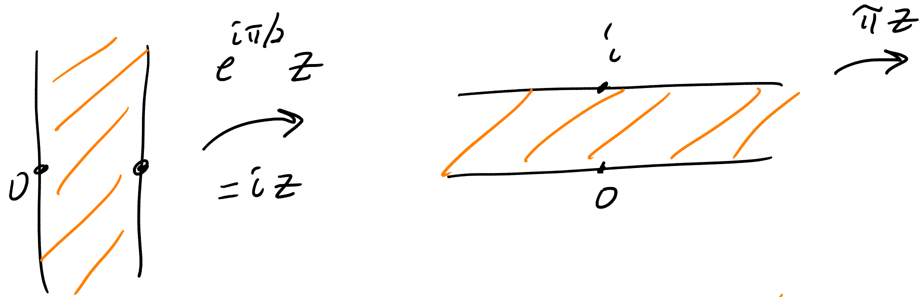
$$\frac{d}{dz} = \frac{1}{(z^2+4z+1)} \cdot \left[2z+4\right]$$ chain rule.

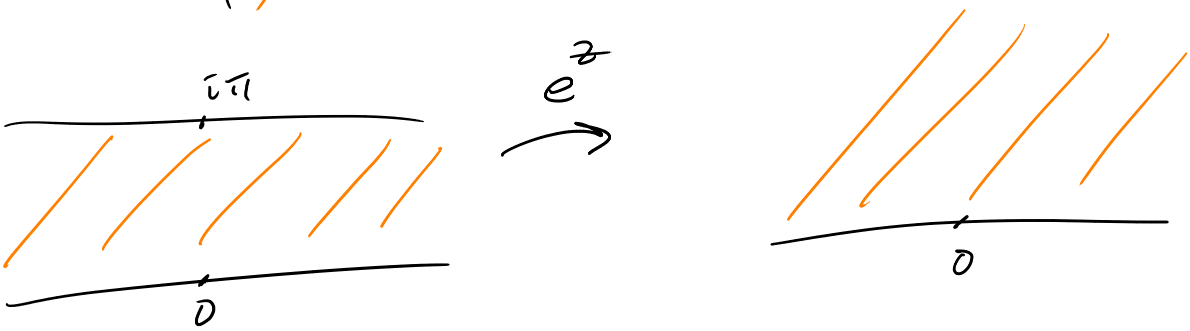
$$\frac{d}{dw}\log_\alpha w = \frac{1}{w}. \quad w = z^2+4z+1.$$

14. Find a one-to-one analytic function that maps the strip $\{z : 0 < \operatorname{Re} z < 1\}$ onto the upper half plane.

Think e^z







$f(z) = e^{i\pi z}$ does it.

9. Compute $\int_0^\pi e^{3it}\,dt$ where t is a real variable.

$$\int_0^{\pi} \frac{d}{dt}\left[\frac{1}{3i} e^{3it}\right] dt = \left[\frac{1}{3i} e^{3it}\right]_0^{\pi}$$

$$f(z) = e^{z} \quad \text{where} \quad z = 3it$$

$$\frac{d}{dt} = f'(z(t))\,z'(t) = e^{[z(t)]}\,z'(t)$$

$$= e^{3it} \cdot 3i$$