Lecture 21 Power series, cont'd

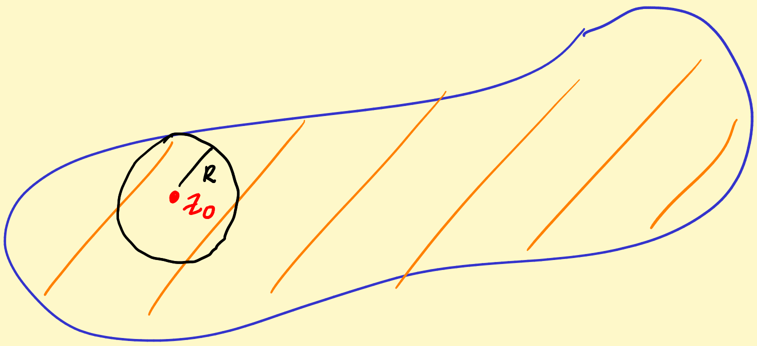
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\sinh z = \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\cosh z = \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$



f on Ω analytic

Cauchy integral formula showed that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

converges absolutely for $z \in D_R(z_0)$ and uniformly on $D_r(z_0)$ where $0 < r < R$. [So R. of C. of series is $\geq R$.]

Easiest way to see power series have a radius of convergence.

Take $z_0 = 0$. If $\sum_0^{\infty} a_n w^n$ converges at w, then we know that $a_n w^n \to 0$ as $n \to \infty$. Take $\varepsilon = 1$ in def^n of limit; Get $N \in \mathbb{N}$ such that

$|a_n w^n| < \underbrace{1}_{\varepsilon}$ when $n > N$.

If z is such that $|z| < |w|$, then

$$|a_n z^n| = \left| a_n z^n \cdot \frac{w^n}{w^n} \right| = \left| a_n w^n \left(\frac{z}{w}\right)^n \right|$$

$$= |a_n w^n| \left|\frac{z}{w}\right|^n < 1 \cdot \left|\frac{z}{w}\right|^n \text{ when } n > N.$$

Aha! Tail end of $\sum_0^\infty a_n z^n$ can be compared to tail end of convergent geometric series $\sum_0^\infty \underbrace{\left|\frac{z}{w}\right|}_{<1}^n$.

Get <u>absolute</u> convergence in $D_r(0)$ where $r = |w|$.

Same idea yields <u>uniform</u> convergence on $D_\rho(0)$ where $\rho < |w|$.

<u>Note:</u> Showed that if $|a_n r^n| < M$ for $n > N$, then $\sum a_n z^n$ converges for $|z| < r$.

Also, if $|a_n r^n|$ <u>cannot</u> be bounded for $n > N$, then $a_n r^n \not\to 0$ and $\sum_0^\infty a_n z^n$ couldn't converge if $|z| = r$.

<u>Fact</u>: R. of C. $R = \sup\{r : |a_n r^n| \text{ is a bounded seq}\}$.

<u>EX</u>: $\sum_{n=1}^\infty z^{n!} = z + z^2 + z^6 + z^{24} + \cdots$

Hmmm. $a_n = 0$ or 1. So $|a_n r^n| = \begin{cases} 0 \\ r^n \end{cases} < 1$ when $0<r<1$.

So $R \geq 1$. What about $r=1$: $|a_n 1^n| = \begin{cases} 0 & n \neq m! \\ 1 & n = m! \end{cases}$

Terms $a_n z^n$ <u>don't</u> go to zero as $n \to \infty$ when $|z|=1$.

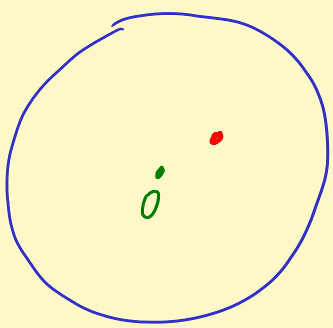
So $R \leq 1$ too because series diverges on $C_1(0)$.

<u>EX</u> $\sum_{n=0}^{\infty} a_n z^n$ where $a_n = p_n + i q_n$ where these are an ennumeration of the points in $D_1(0)$ with rational real and imag parts.

Same reasoning as for $\sum z^{n!}$ yields $R \geq 1$.

Think about convergence at $z=1$: $\sum_0^\infty a_n$ conv?

No!



$\bullet = \frac{1}{\pi} + i\frac{1}{e}$ (irrational Re, Im parts)

Get arbitrary large n with $p_n \approx \frac{1}{\pi}$, $q_n \approx \frac{1}{e}$! So $a_n \not\to 0$.

$R \leq 1$ too.

<u>EX</u>: $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

Ratio test shows $R=1$.

Converges on $D_1(0)$.

Diverges when $|z|>1$.

Converges when $|z|=1$ by comparison

$$\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2} \quad \text{convergent } p\text{-series } p=2$$

Converges uniformly on $\overline{D_1(0)}$! (Special!)

EX: $\sum_{1}^{\infty} \frac{z^n}{n}$ has R. of C. $= 1$, but it diverges

at $z=1$. What about $z=-1$? (Alternating series test)

EX $f(z) = \text{Log } z$ (principal branch)

$$= \sum_{n=0}^{\infty} a_n (z-1)^n$$

$f(z) = \text{Log } z$ $\qquad$ $f(1) = \text{Log } 1 = \ln 1 + i \cdot 0 = 0$

$f'(z) = \frac{1}{z}$ $\qquad$ $f'(1) = 1$

$f''(z) = \frac{-1}{z^2}$ $\qquad$ $f''(1) = -1$

$f'''(z) = \frac{2}{z^3}$ $\qquad$ $f'''(1) = 2!$

$f^{(4)}(z) = \frac{-3 \cdot 2}{z^4}$ $\qquad$ $f^{(4)}(1) = -3!$

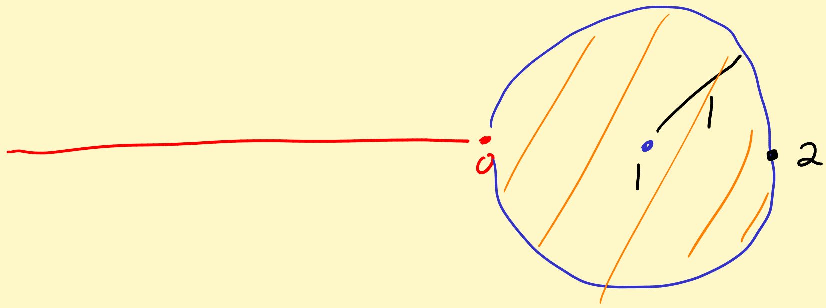
$f^{(5)}(1) = 4! = (-1)^{n+1}(n-1)!$

$\vdots$

$$\frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n+1}(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$$

$$\text{Log } z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \cdots$$

$$= \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$



Cauchy integral formula yields $R \geq 1$.

Think: If power series converged on $D_\rho(1)$ with $\rho > 1$, it would converge to analytic $F(z)$ on $D_\rho(1)$.

But $F(z) = \text{Log}\, z$ in $D_1(1)$.

doesn't blow up!

blows up as $z \to 0$

So $R \leq 1$. [Ratio test shows this too.]

Fact: We know power series conv uniformly on $D_r(z_0)$ when $0 < r < R = $ R.of C. Last time: So Morera's $\Rightarrow$ $f(z) =$ power series is analytic inside $D_R(z_0)$. a_n are uniquely determined by Taylor's formula $a_n = \dfrac{f^{(n)}(z_0)}{n!}$. [Power series expansions are unique!]

<u>Big fact</u> Can differentiate power series term by term.
Radius of conv stays same!