Lecture 22 Hadamard's formula

R. of C. of $\sum_{n=0}^{\infty} a_n z^n$

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

$$= \lim_{N\to\infty} \underbrace{\text{Sup}}\left\{ \sqrt[n]{|a_n|} : n \geq N \right\}$$

ok for $R=0$, or $R=\infty$

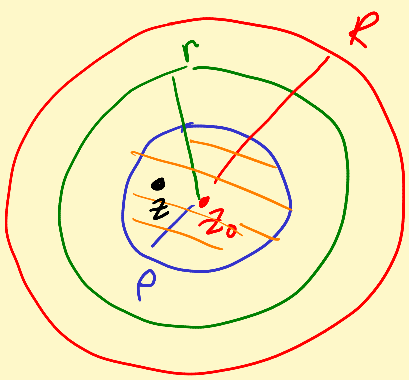
"least upper bound"

Note: $\left\{ \sqrt[n]{|a_n|} : n \geq N \right\}$ get smaller and smaller as N grows,

so Sup's can only <u>decrease</u>.

Non-increasing seq bounded below by zero has a limit.

<u>Think</u>: Those Sup's are sliding <u>down</u> to $\frac{1}{R}$.



$z_0 = 0$

$0 < \rho < r < R$

<u>Why</u> Hadamard's works:

$$\frac{1}{R} < \frac{1}{r} < \frac{1}{\rho}$$

$\text{Sup}\left\{ \sqrt[n]{|a_n|} : n \geq N \right\}$ sliding down to $\frac{1}{R}$

So, there is an N such that

$$\underbrace{\sup}_{\text{least upper bound}}\{\sqrt[n]{|a_n|} : n \geq N\} < \underset{\substack{\uparrow \\ \text{upper bound}}}{\frac{1}{r}}$$

So $\sqrt[n]{|a_n|} < \frac{1}{r}$ for $n \geq N$

Take n-th power: $|a_n| < \frac{1}{r^n}$

Multiply by $|z|^n$: $|a_n z^n| < \left(\frac{|z|}{r}\right)^n < \left(\frac{\rho}{r}\right)^n$

Aha! Tail end of $\sum_0^\infty a_n z^n$ compares to convergent geom series $\sum_{n=N}^\infty \left(\frac{\rho}{r}\right)^n$

$\rho/r < 1$.

[Can also deduce by comparison that partial sums $\sum_0^m a_n z^n$ are <u>uniformly Cauchy</u> on $D_\rho(0)$.

So they converge uniformly on $D_\rho(0)$.]

<u>Last step</u> Show $\sum a_n z^n$ <u>diverges</u> when $|z| > R$:

$$\frac{1}{|z|} \underset{\substack{\uparrow \\ <!}}{<} \frac{1}{R} \leq \sup\{\sqrt[n]{|a_n|} : n \geq N\}.$$

So, for *each* N, there is a $n \geq N$ such that

$$\frac{1}{|z|} < \sqrt[n]{|a_n|}$$

Take n-th power: $\frac{1}{|z|^n} < |a_n|$

$$1 < |a_n z^n| \leftarrow \text{get } n \geq N \text{ for each } N.$$

Aha! Terms $a_n z^n$ *do not* tend to zero as $n \to \infty$.

Series diverges. R. of C. $= \infty$.

EX: $\sum_{n=1}^{\infty} z^{n!} = z + z^2 + z^6 + z^{24} + \cdots$

$$a_n = \begin{cases} 1 & n = m! \\ 0 & \text{otherwise} \end{cases}$$

$\text{Sup}\{\sqrt[n]{|a_n|} : n \geq N\} = 1.$

$\frac{1}{R} = \text{Lim Sup} = 1.$

So $R = 1$. (as before).

Advice: Use Ratio test (or root test) if *you can* instead of Hadamard.

Note: Knowing power series converge *uniformly* on $D_r(z_0)$ where $0 < r < \text{R. of C.}$ means we can use Morera's + Cauchys thm on a disc to deduce that power series converge to analytic fcns.

Consequence: Power series can be differentiated term by term.

R. of C. = R $\quad f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

Same R. of C. $\longrightarrow$ $f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots$

$$f''(z) = 2a_2 + 3\cdot 2a_3(z-z_0) + \cdots$$

$$\vdots$$

Let $z=z_0$. See $a_n = \dfrac{f^{(n)}(z_0)}{n!}$. Taylor's

Fun thing: Compute $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$f(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z} \qquad |z|<1 \qquad R=1$$

Take $f'(z) = 1 + 2z + 3z^2 + \cdots = \dfrac{1}{(1-z)^2} \qquad |z|<1 \quad (R=1)$

$$zf'(z) = z + 2z^2 + 3z^3 + \cdots = \frac{z}{(1-z)^2} \qquad |z|<1.$$

Let $z=\frac{1}{2}$: $\displaystyle\sum_{1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-\frac{1}{2})^2} = 2$

Can solve ODE's with power series.

$$y'' - x y' - y = 0 \qquad y(x)$$

Let $x = z \in \mathbb{C}$.

$$f''(z) - z f'(z) - f(z) = 0$$

Hope sol^n $f(z)$ is analytic near $z=0$.

Try $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Then $$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$$

multiply $f'(z)$ by z

Plug into ODE $$\underbrace{\sum_{2}^{\infty} n(n-1) a_n z^{n-2}}_{\sum_{0}^{\infty} (n+2)(n+1) a_{n+2} z^n} - \sum_{n=1}^{\infty} n a_n z^n - \sum_{n=0}^{\infty} a_n z^n$$

$$\underbrace{\left(2\cdot 1\cdot a_2 - a_0\right)}_{n=0} + \underbrace{\left(3\cdot 2 a_3 - 2a_1\right)}_{n=1} + \sum_{n=2}^{\infty}\Big[\underbrace{(n+2)(n+1)a_{n+2} - (n+1)a_n}_{=0}\Big] z^n = $$

want

$$\equiv 0$$

Power series $\equiv 0$ only when all coeff $=0$. (Taylor's).

$n=0$: $a_2 = \frac{1}{2} a_0$ ✓

$n=1$: $a_3 = \frac{1}{3} a_1$ ✓

$n=2$: $a_4 = \frac{1}{4} a_2 \longleftarrow$ $a_4 = \frac{1}{4} a_2 = \frac{1}{4\cdot 2} a_0$

$a_5 = \frac{1}{5} a_3 = \frac{1}{5\cdot 3} a_1$

$\vdots$

n : $\boxed{a_{n+2} = \frac{a_n}{n+2}}$ Recursion formula : $n \geq 2$.

Note: $f(0) = a_0$, $f'(0) = a_1 \longleftarrow$ initial conditions

Get rest from recursion.

n even: $a_n = \frac{1}{n(n-2)(n-4)\cdots 2} = \frac{1}{n\cdots 6\cdot 4\cdot 2}$

$a_{2n} = \frac{1}{2^n n!}$

Get $f(z) = a_0 + a_1 z + \frac{a_0}{2} z^2 + \frac{a_1}{3} z^3 + \frac{a_0}{4\cdot 2} z^4 + \frac{a_1}{5\cdot 3} z^5 + \cdots$

$$= a_0\left(1 + \frac{z^2}{2} + \frac{z^4}{4\cdot 2} + \frac{z^6}{6\cdot 4\cdot 2} + \frac{z^8}{8\cdot 6\cdot 4\cdot 2} + \cdots\right)$$

$$+ a_1\left(z + \frac{z^3}{3} + \frac{z^5}{5\cdot 3} + \frac{z^7}{7\cdot 5\cdot 3} + \cdots\right)$$

$$= a_0 y_1 + a_1 y_2 \longleftarrow$$ lin ind sol^{n}s. gen^l sol^n

Ratio test: Hmmm. $$\left|\frac{a_{n+2} z^{n+2}}{a_n z^n}\right| = \underbrace{\left|\frac{a_{n+2}}{a_n}\right|}_{\frac{1}{n+2}} |z|^2$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

$$< 1$$

Converges for all z. $R = \infty$.

Found an entire sol^n! Let $z = x$ again.