Lecture 23 Zeroes of analytic functions

Exam 1 ave 79.

A ≥ 80
B ≥ 65
B- ≥ 60
C > 50

Hate the formula $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ when this exists.

Better to use Ratio test directly every time.

Better way: $f(z) = a_0 + a_3 z^3 + a_6 z^6 + a_9 z^9 + \cdots$

where: $a_0 = 1$, $a_{n+3} = \frac{(n+1)(13n+8)^2}{n^3+2n^2+7} a_n$

Ratio test: $L = \lim_{\substack{\text{way out}\\ \to \text{out}}} \left| \frac{\text{Term way out}}{\text{Term before}} \right| \leftarrow \left| \frac{u_{n+1}}{u_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{a_{n+3} z^{n+3}}{a_n z^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)(13n+8)^2}{n^3+2n^2+7} |z|^3$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})(13+\frac{8}{n})^2}{1+\frac{2}{n}+\frac{7}{n^3}} |z|^3 \qquad \frac{(\frac{1}{n^3})}{(\frac{1}{n^3})}$$

$$= (13)^2 |z|^3 \quad \begin{matrix} < 1 & \text{conv} \\ > 1 & \text{div} \end{matrix}$$

R of C $\quad R = \left(\frac{1}{13^2}\right)^{1/3} = 13^{-2/3}$

Order: Ratio test, Root test, Hadamard's, desperation, ...

<u>Zeroes of analytic fcns</u> Feeling: Analytic fcns are like "polys of infinite degree". Zeroes are <u>isolated</u> and can be "factored out".

Basic idea easy: $\quad f(z) = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots$

$$f(z_0) = a_0 = 0 \quad \leftarrow \quad z_0 \text{ zero of order } 1$$

$$\frac{f'(z_0)}{1!} = a_1 = 0 \quad \leftarrow \qquad 2$$

$$\frac{f''(z_0)}{2!} = a_2 = 0 \quad \leftarrow \qquad 3$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{f^{(N-1)}(z_0)}{(N-1)!} = a_{N-1} = 0 \quad \leftarrow \qquad N$$

$$\frac{f^{(N)}(z_0)}{N!} = a_N \neq 0$$

← First non-zero coeff in Taylor series

$$f(z) = a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots \qquad R = R \text{ of } C > 0$$

$$= (z-z_0)^N \left[\underbrace{a_N + a_{N+1}(z-z_0) + a_{N+2}(z-z_0)^2 + \cdots}_{F(z)} \right]$$

This series converges for exactly the same z as one for $f(z)$!

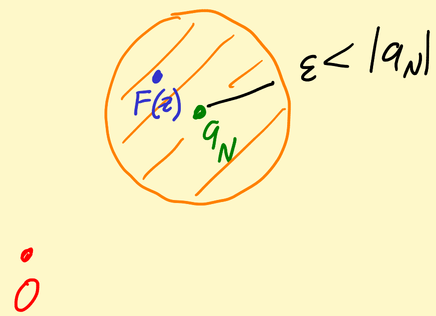
$[(z-z_0)^N$ is just a complex # for z fixed.$]$

Conclude: Power series for $F(z)$ has same R of C as one for $f(z)$.

f, F analytic on $D_R(z_0)$. $\quad f(z) = (z-z_0)^N F(z)$

<u>and</u> $\quad F(z_0) = a_N \leftarrow \neq 0$

Zero of f at z_0 is <u>isolated</u>.



F analytic $\Rightarrow$ F continuous.

Let $\varepsilon < |a_N|$.

$|F(z) - a_N| < \varepsilon$

when $|z - z_0| < \delta$.

So $F(z) \neq 0$ when $|z-z_0| < \delta$.

Hence, the <u>only</u> zero of $f(z)$ in $D_\delta(z_0)$ is $z = z_0$.

Def^n of an isolated zero.

Def^n If a_N is the first non-zero coeff of Taylor series for $f(z)$ at z_0, say $f(z)$ has a zero of order N (or multiplicty N) at z_0.

Thm: $f(z) = (z-z_0)^N \underbrace{F(z)}$

same R of C as f series.

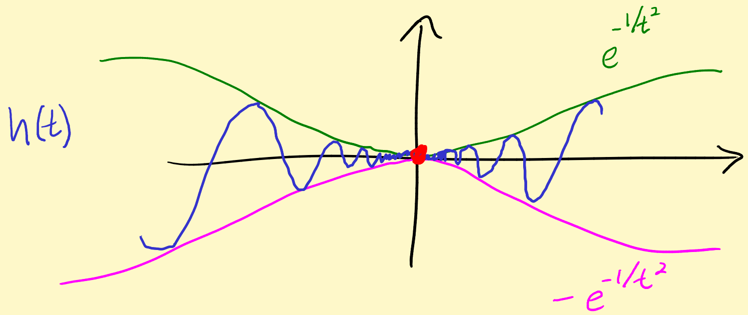
$F(z_0) \neq 0$.

$\Longleftrightarrow$ f has a zero of order N at z_0.

Way false $\mathbb{R} \to \mathbb{R}$

$$h(t) = \begin{cases} 0 & t=0 \\ e^{-1/t^2} \sin\frac{1}{t} & \end{cases}$$

C^∞-smooth on $\mathbb{R}$.



Zeroes pile up at $t=0$. Not $\equiv 0$.

$t_0=0$. Taylor series for $h \equiv 0 \neq h(t)$ near $t_0=0$.

Big Theorem: Identity thm. Suppose f analytic on a domain Ω. If the zeroes of "pile up" at a point z_0 in Ω, then $f \equiv 0$ on Ω.