Lesson 24 More about power series

HWK 7 due Wed, March 20, 11:30 pm in Gradescope. [5.5 problems removed]

Famous power series: $a_0 = 1$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$

1, 1, 2, 3, 5, 8, ... Fibonacci #'s

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

What is the R. of C.?

Tricky, but nice method. Assume $R > 0$. [Don't know that yet!]

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = 1 + 1\cdot z + \sum_{2}^{\infty} a_n z^n$$

$$z f(z) = \sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{m=1}^{\infty} a_{m-1} z^m = \sum_{n=1}^{\infty} a_{n-1} z^n$$

shifting indices: Let $m = n+1$
$n = m-1$

when $n=0$: $m=1$
as $n \to \infty$: $m \to \infty$

$$z^2 f(z) = \sum_{n=0}^{\infty} a_n z^{n+2} = \sum_{n=2}^{\infty} a_{n-2} z^n$$

Aha! $$f(z) - z f(z) - z^2 f(z) = (1+z) - (z) + \sum_{n=2}^{\infty} \underbrace{\left[a_n - a_{n-1} - a_{n-2}\right]}_{0} z^n$$
$$= 1$$

$$f(z) = \frac{1}{1 - z - z^2} \qquad \text{Wow!}$$

R = R. of C. of power series about $z_0 = 0$ is the distance to nearest zero of denom to 0.

Note: Can now go backwards and see that Taylor coeff for f at $z=0$ are indeed the F. #'s.

Roots $\quad z^2+z-1=0$

$$z=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}$$

closest to zero: $\dfrac{\sqrt{5}-1}{2}$ $\leftarrow$ Golden ratio!

p. 267: 3. a) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{3^n+4^n}z^n$

Ratio test? $\displaystyle\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right|=\frac{\left(\frac{2^{n+1}}{3^{n+1}+4^{n+1}}\right)}{\left(\frac{2^n}{3^n+4^n}\right)}|z|$

$$=2\cdot\frac{3^n+4^n}{3^{n+1}+4^{n+1}}|z|\cdot\frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{4}\right)^n}$$

$$=2\,\frac{\left(\frac{3}{4}\right)^n+1}{3\left(\frac{3}{4}\right)^n+4}|z|$$

$$\longrightarrow 2\cdot\frac{0+1}{3\cdot 0+4}|z| \quad \begin{array}{l} <1 \ \text{conv} \\ >1 \ \text{div.}\end{array}$$

b) $\displaystyle\sum_{n=0}^{\infty}2^n z^{n^2}$

Hadamard? Ratio?

Ratio: $\displaystyle\left|\frac{2^{n+1}\,z^{(n+1)^2}}{2^n\,z^{n^2}}\right|=2|z|^{2n+1}$ Aha! $\longrightarrow 0$ if $|z|<1$.

<1

So conv. if $|z|<1$.

What if $|z| \geq 1$? $\left|2^n z^{n^2}\right| = 2^n \underbrace{|z|^{n^2}}_{\geq 1} \geq 2^n$.

terms don't $\to 0$. It diverges! $R=1$.

Hadamard? a_{n^2} = term in front of z^{n^2}

$= 2^n$

$$\sqrt[n^2]{|a_{n^2}|} = \left(2^n\right)^{1/n^2} = \left(\left[2^n\right]^{1/n}\right)^{1/n} = 2^{1/n}$$

$$\to 1.$$

Fact. $\lim\limits_{n\to\infty} r_n = L \iff \liminf\limits_{n\to\infty} r_n = L = \limsup\limits_{n\to\infty} r_n$

So $\dfrac{1}{R} = \limsup\limits_{n\to\infty} \sqrt[n^2]{|a_{n^2}|} = \underline{1}$. $R=1$ ✓

c) $$\sum_{n=0}^{\infty} \left[2+(-1)^n\right]^n z^n = \underbrace{3^1 + z + 3^2 z^2 + z^3 + 3^3 z^4 + z^5 + \cdots}_{\text{pull apart}}$$

R of C = min R of two.

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{N\to\infty} \underbrace{\sup_{n\geq N} \left\{1, \left(3^n\right)^{1/n}\right\}}_{=3}$$

3 e) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n} z^{2n}$

Ratio test. ✓

Book: $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ when that limit exists.

Could use: Let $w = z^2$. $\sum a_n w^n$.

Get R. of C. $|w| < R$

$|z^2| < R$

$|z| = \sqrt{R} \leftarrow$ R of C.

f) $\displaystyle\sum_{n=0}^{\infty} z^{n!}$ Hadamard!