Lesson 25 The Uniqueness theorem, Laurent expansions

first non-zero coeff in Taylor series at z_0.

$$\underbrace{f(z_0)}_{a_0=0}=0 : \quad f(z) = a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + a_{N+2}(z-z_0)^{N+2} + \cdots$$

$$= (z-z_0)^N \left[a_N + a_{N+1}(z-z_0) + a_{N+2}(z-z_0)^{N+2} + \cdots \right]$$

same R of C as power series above!

$$= (z-z_0)^N F(z), \qquad F(z_0) = a_N \neq 0$$

Thm: If $f \not\equiv 0$, then zeroes of f are isolated.

Lemma If f is analytic on $D_r(z_0)$ and $f(z_0)=0$, then, either

A) $f \equiv 0$ on $D_r(z_0)$, or

B) the zero at z_0 is isolated.

Def^n: z_0 is an accumulation point (or limit point) of a set $S' \subset \mathbb{C}$, if there is a seq $\{z_n\}_{n=1}^{\infty}$ with $z_n \in S'$, $z_n \neq z_0$ for all n, and $z_n \to z_0$ as $n \to \infty$. [or cluster point]

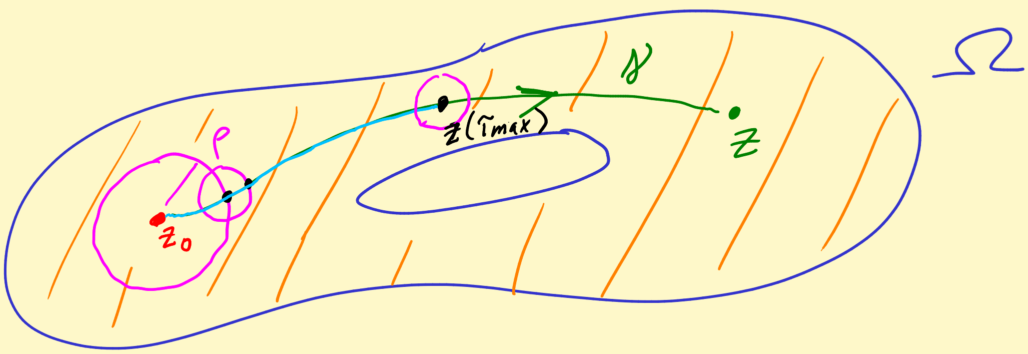
equivalent to: For every $\varepsilon > 0$, there is a point z in $D_\varepsilon(z_0) \cap S'$ with $z \neq z_0$.

Big fact: (Uniqueness theorem). Suppose f is analytic on a domain Ω and let $Z_f = \{z \in \Omega : f(z) = 0\}$.

If Z_f has an accumulation pt in Ω, then $f \equiv 0$ on Ω.

Last time examples: Way false $\mathbb{R} \to \mathbb{R}$.

Pf:



Suppose $z_0 \in \Omega$ is a limit pt of Z_f. $\exists\, D_\rho(z_0) \subset \Omega$ open.

Lemma implies $f \equiv 0$ on $D_\rho(z_0)$. Pick $z \in \Omega$.

Ω connected. So $\exists\, \gamma: z(t),\ a \le t \le b$ in Ω connecting z_0 to z. Hmmm. $f(z(t)) \equiv 0$ for $t \in [a, a+\delta)$ for some $\delta > 0$.

Let $\tau_{max} = \sup\{\tau \in [a,b]$ such that $f(z(t)) = 0$ for $a \le t < \tau\}$

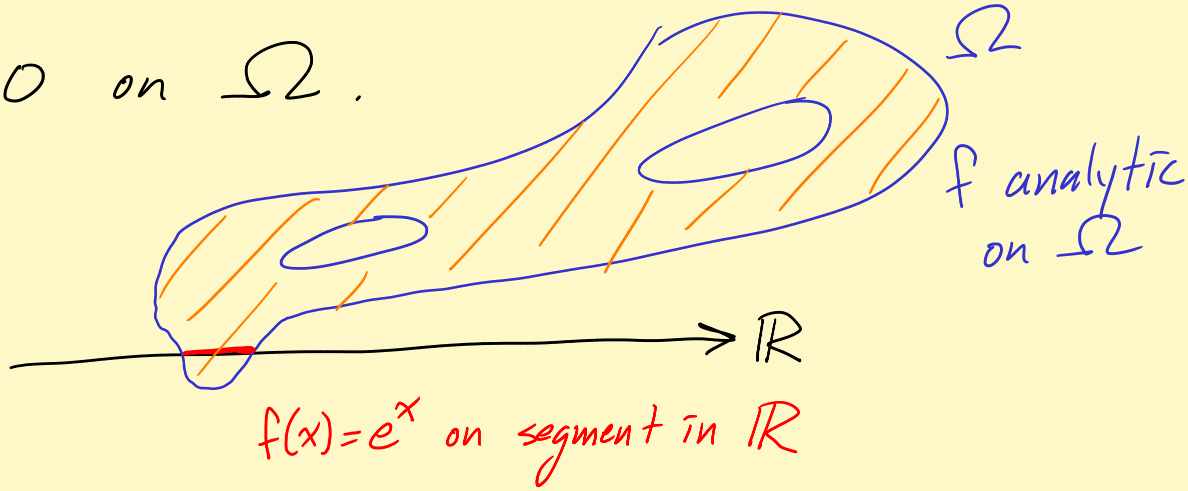
Aha! $z(\tau_{max})$ is also a limit pt of Z_f in Ω.

$\exists\, D_r(z(\tau_{max})) \subset \Omega$, $r > 0$. Lemma $\Longrightarrow$ $f \equiv 0$ on $D_r(z(\tau_{max}))$. Oops! Could have made τ_{max} bigger? No. Only way this can happen is if $\tau_{max} = b$. Continuity implies

$f(z) = \lim_{t \to b^-} \underbrace{f(z(t))}_{0} = 0.$ z was <u>arbitrary</u>.

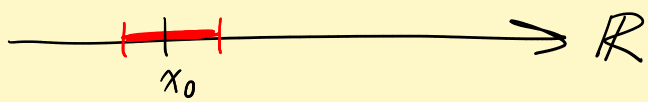
So $f \equiv 0$ on Ω.

<u>EX</u>:



Then $f(z) = e^z$ on Ω.

In particular, e^z is the only way to extend e^x from $\mathbb{R}$ to $\mathbb{C}$ as an analytic fcn.



Every pt $x_0 \in$ segment is a limit pt of segment in $\mathbb{C}$.

Apply Uniq thm to $f(z) - e^z$

<u>Fun fact</u>: Every trig identity holds for <u>complex</u> angles.

<u>EX</u>: $\sin 2\theta = 2 \sin\theta \cos\theta$ $\qquad \left[\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \right]$

$$\sin(\alpha + \beta) = \cos\alpha \sin\beta + \cos\beta \sin\alpha$$

Let $\alpha = z$ go complex. Get extension to $z \in \mathbb{C}$. Fix z.

Next, let $\beta = w$ go complex.

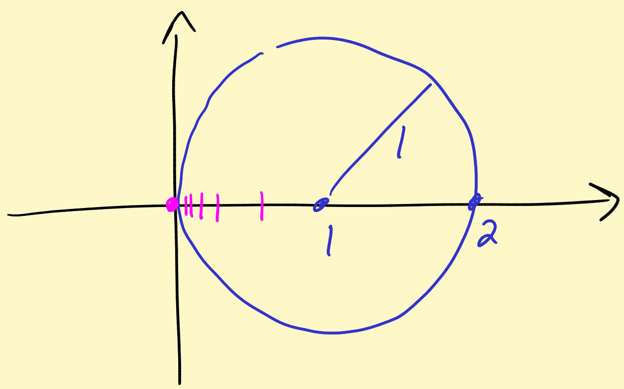
<u>Warning</u> Limit pt z_0 needs to be <u>in</u> Ω.

<u>EX</u>: $f(z) = \sin \frac{1}{z}$ on $\Omega = D_1(1)$.

Zeroes of $\sin w$: $w = n\pi$, $n \in \mathbb{Z}$.

$\frac{1}{z} = n\pi$. $z = 1/n\pi$

$$Z_f = \{ z \in D_1(1) : \sin \tfrac{1}{z} = 0 \} = \underbrace{\{ \tfrac{1}{\pi}, \tfrac{1}{2\pi}, \tfrac{1}{3\pi}, \dots \}}_{0 \text{ is a limit pt}}$$



$f(z) \not\equiv 0$.

0 is <u>not</u> <u>in</u> Ω,

it's in the boundary of Ω.

<u>Cor</u> If f is analytic on a domain Ω and f vanishes on $D_\varepsilon(z_0) \subset \Omega$, then $f \equiv 0$ on Ω.

[$z_0 \in \Omega$ is a limit pt of Z_f.]

<u>Cor</u>: Same is true for <u>harmonic</u> fcns u.

<u>Pf</u>: Know $u_x - i u_y$ is <u>analytic</u> when u is harmonic <u>on</u> Ω. Let $F = u_x - i u_y$ on Ω. Then $F \equiv 0$

on $D_\varepsilon(z_0)$. So Cor $\Rightarrow$ $F \equiv 0$ on Ω.

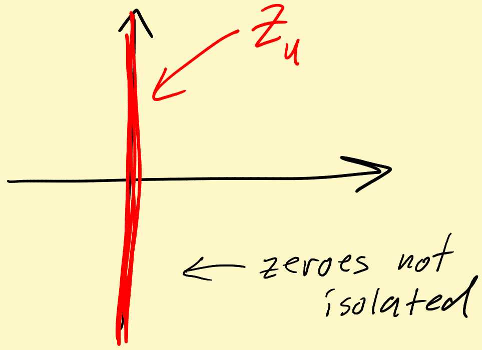
So $u_x \equiv 0$, $u_y \equiv 0$ on Ω, i.e., $\nabla u \equiv 0$.

So u is <u>constant</u> on Ω. Since $u(z_0) = 0$, that constant must be zero.

<u>Question</u> Is there is a uniquess thm for harmonic fcns? No!

<u>EX</u>: $x = \mathrm{Re}\, z$ is harmonic.

$u(x,y) = x$



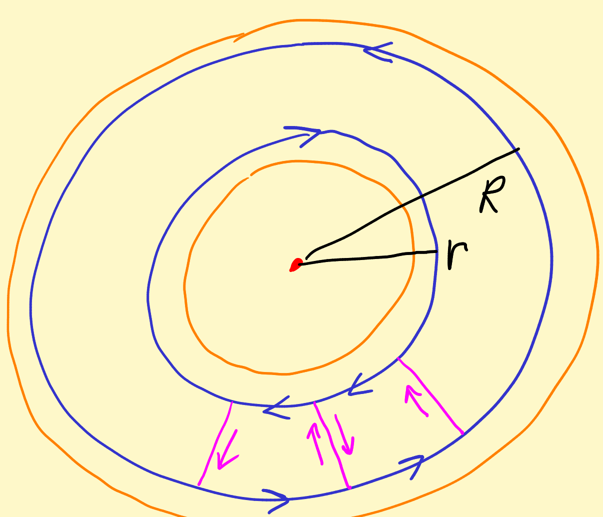
Lots of limit pts of zeroes: imaginary axis $= Z_u$.

Next time: Laurent expansions.

Tool: Cauchy thm and integral formula for an annulus.

Cauchy thm

$$\left(\int_{C_R} - \int_{C_r}\right) f\, dz = 0$$



f analytic in bigger annulus

<u>Pf</u>: Cut up like a pizza. Add up integrals. Cuts cancel!