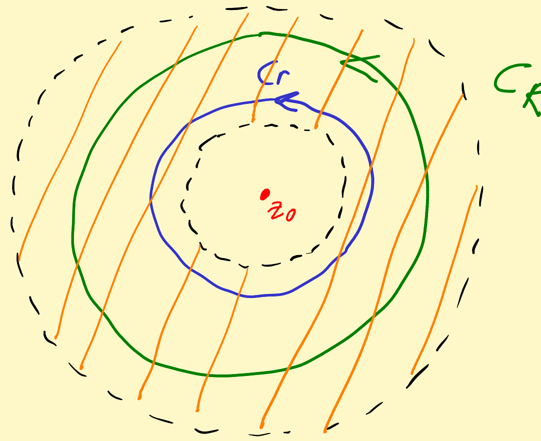
Lesson 26 Laurent expansions, Cauchy thm for an annulus

Cauchy thm for an annulus

f analytic on a bigger annulus.

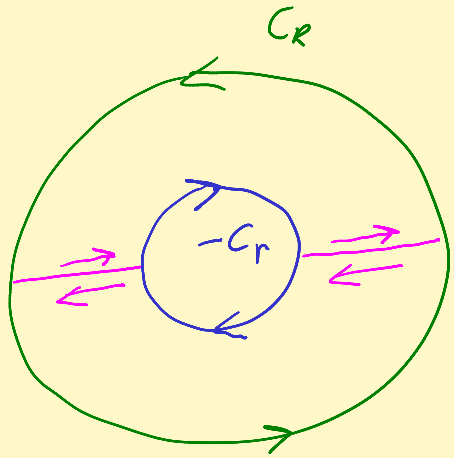


$0 < r < R$

$$\left(\int_{C_R} - \int_{C_r}\right) f\,dz = 0$$

both C_r, C_R are counterclockwise circles

why:



With cuts as shown, see

$$\left(\int_{C_R} - \int_{C_r}\right) f\,dz$$

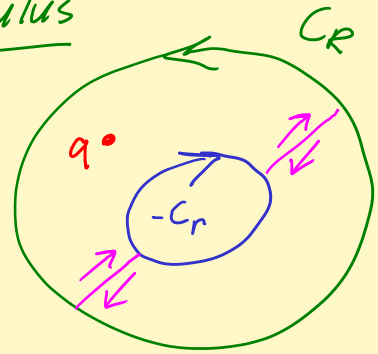
$$= \left(\int_{C_R} + \int_{-C_r}\right) f\,dz$$

$$= \left(\underbrace{\int_{Top} f\,dz}_{=0} + \underbrace{\int_{Bottom} f\,dz}_{=0}\right)$$

by Cauchy's thm.

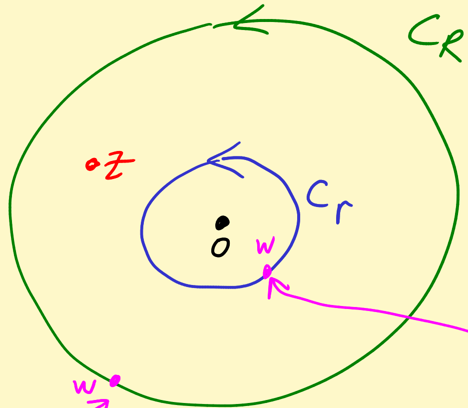
Cauchy integral formula for an annulus

$$f(a) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right)\frac{f(z)}{z-a}\,dz$$



$$= \frac{1}{2\pi i}\left(\underbrace{\int_{\text{Left half}}}_{\substack{2\pi i f(a) \\ \text{by Cauchy} \\ \text{integral formula}}} + \underbrace{\int_{\text{right half}}}_{\substack{=0 \\ \text{by Cauchy's} \\ \text{thm}}}\right)\frac{f(z)}{z-a}\,dz$$

Laurent expansions



$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z}\,dw}_{\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w}\underbrace{\frac{1}{1-\frac{z}{w}}}_{1+\left(\frac{z}{w}\right)+\left(\frac{z}{w}\right)^2+\cdots+\left(\frac{z}{w}\right)^N+E_N\left(\frac{z}{w}\right)}dw} - \underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dw}_{\text{new stuff}}$$

$$= \lim_{N\to\infty}\sum_{n=0}^{N}\underbrace{\left(\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w^{n+1}}\,dw\right)}_{a_n} z^n$$

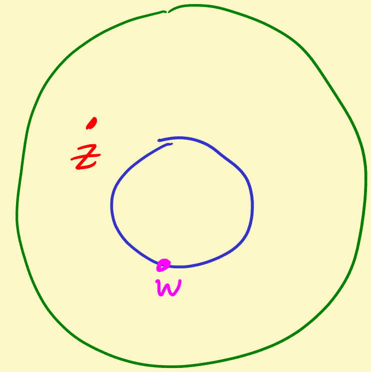
(a_n: Taylor's, *if* f analytic inside C_R)

old stuff gives a power series that converges inside $C_R(0)$.

So R of C $\geq$ R.

New stuff

$$-\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-z}\,dw$$



$$= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{z}\underbrace{\frac{1}{1-\frac{w}{z}}}_{1+\left(\frac{w}{z}\right)+\left(\frac{w}{z}\right)^2+\cdots+\left(\frac{w}{z}\right)^N+E_N\left(\frac{w}{z}\right)}\,dw$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C_r} f(w)\,w^{n-1}\,dw\right)\frac{1}{z^n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w^{(-n)+1}}\,dw\right) z^{-n}$$

Wow! Same formula as old stuff, but with $-n$ in place of $+n$.

Get $$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n z^n}_{} + \underbrace{\sum_{n=1}^{\infty} a_{-n} z^{-n}}_{}$$

converges absolutely and uniformly on $D_\rho(0)$ for every $\rho < R$. R of C $\geq$ R

converges absolutely and uniformly on $\{z : |z| > \rho\}$ for any ρ with $\rho > r$.

Second sum = power series $\sum_{n=1}^{\infty} a_{-n} w^n$ with R of $C > \frac{1}{r}$,

think $w = \frac{1}{z}$.

Convention in complex analysis: $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ means

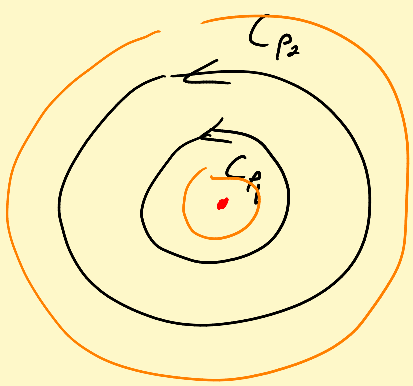
$\sum_0^{\infty}$ and $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ both converge.

Cool thing: $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ where $a_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{w^{n+1}}\, dw$

and $r \le \rho \le R$.

<u>Note</u>: Cauchy thm for annulus shows

G analytic on bigger annulus



$\rho_1 < \rho_2$

$$\left(\int_{C_{\rho_2}} - \int_{C_{\rho_1}} \right) G(w)\, dw = 0$$

So $\int_{C_{\rho_2}} = \int_{C_{\rho_1}}$

$$G(w) = \frac{f(w)}{w^{n+1}}$$

Put it all together: Laurent expansion converges absolutely and <u>uniformly</u> on $A(\rho_1, \rho_2) = \{ z : \rho_1 < |z - z_0| < \rho_2 \}$

where $r < \rho_1 < \rho_2 < R$.

Case $z_0 \neq 0$. $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

EX: a) $$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z}$$

$$= \underbrace{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}_{\text{converges for same } z \text{ as power series for } \sin z,\ \text{all } z.} = H(z)$$

entire!

Interesting! $\frac{\sin z}{z}$ is not singular at $z=0$.

In fact $$H(z) = \begin{cases} 1 & z=0 \\ \frac{\sin z}{z} & z \neq 0 \end{cases}$$ is entire.

$z=0$ is a "removable singularity".

b) $$\frac{\cos z}{z^3} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z^3}$$

$-\frac{1}{2}$ = "residue at $z=0$"

$$= \frac{1}{z^3} - \frac{1/2}{z} + \frac{1}{4!} z - \frac{1}{6!} z^2 + \cdots$$

Finitely many a_{-n} terms $a_{-n} \neq 0$.

$z=0$ is a "pole" of $\frac{\cos z}{z^3}$. $\lim_{z \to 0} \frac{\cos z}{z^3} = \infty$.

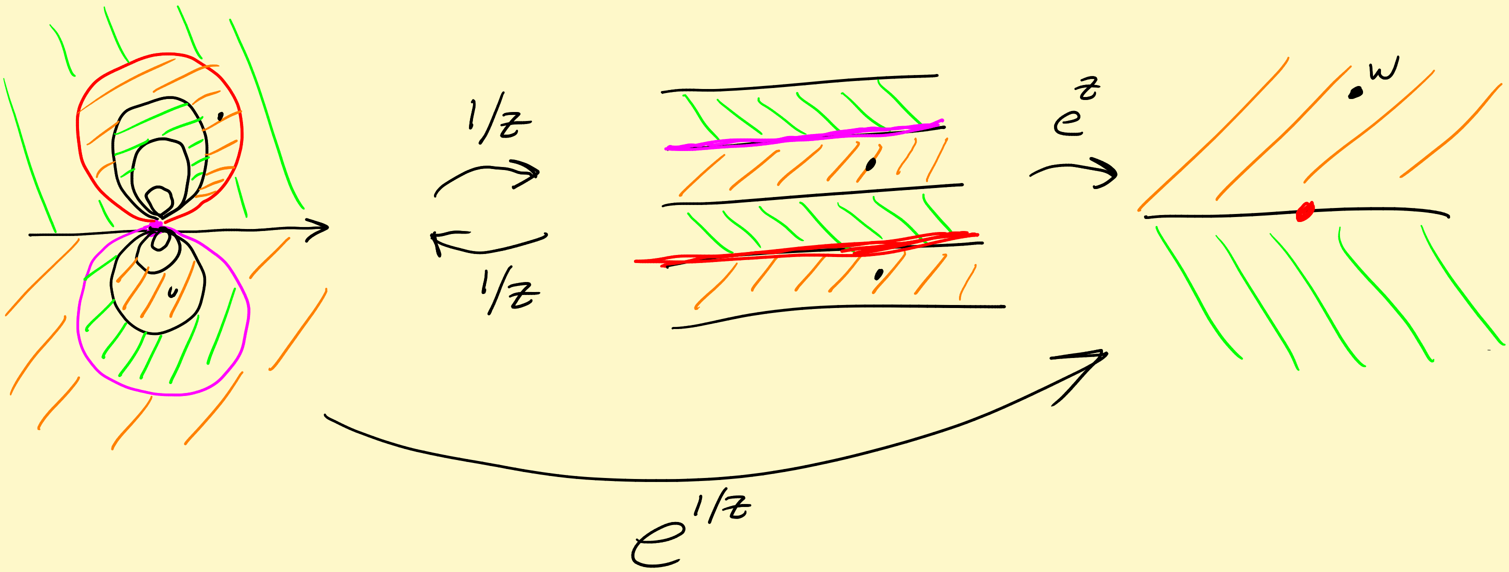
c) $e^w = 1 + w + \frac{w^2}{2!} + \cdots$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1/2!}{z^2} + \frac{1/3!}{z^3} + \cdots$$

Infinitely many a_{-n} with $a_{-n} \neq 0$.

$z=0$ is an "essential singularity".

$e^{1/z}$ <u>shreds</u> the complex plane at $z=0$.



$e^{1/z}$ is <u>infinite-to-one</u> map of

$D_\varepsilon(0) - \{0\}$ onto $\mathbb{C} - \{0\}$, no matter

how small $\varepsilon > 0$ is!